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MATHEMATICS

On the problem of non-local existence for first integrals of a system of ordinary differential equations

by
A. PLIŚ

Presented by T. WAŻEWSKI on December 10, 1954

In the present note we shall give an example of a system of two ordinary differential equations of the form

$$\frac{dy_1}{dx} = Q_1(x, y_1, y_2)$$
$$\frac{dy_2}{dx} = Q_2(x, y_1, y_2), \tag{1}$$

where the functions $Q_i(x, y_1, y_2)$ $(i=1,2)$ are of class C^∞ in the space E_3, the system being such that every function $z(x, y_1, y_2)$, continuous in E_3 and constant along the integrals of that system, is constant throughout the whole space E_3*). (We do not assume the differentiability of the function $z(x, y_1, y_2)$).

1. Let U be a system of ordinary differential equations defined in the region D.

Definition 1. Points A, B are called directly conjugate with respect to system U if there exist sequences of points A_m, B_m, A_m tending to A, and B_m tending to B, such that for every index m the points A_m and B_m lie on the same integral of system U.

Definition 2. Points $\widetilde{P}, \widetilde{\widetilde{P}}$ are called conjugate with respect to system U if there exists a finite sequence of points $P_1, P_2, \ldots, P_k$ such that $P_1 = \widetilde{P}, P_k = \widetilde{\widetilde{P}}$, and for the index $i = 1, 2, \ldots, k-1$ the points P_i and P_{i+1} are directly conjugate with respect to system U.

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*) An analogous subject for the plane is treated by the author in the paper, *The problem of non-local existence for solutions of linear partial differential equation of the first order*, which will shortly appear in the periodical Annales Polonici Mathematici. The paper quotes works dealing with related subjects.

Definition 3. The totality of points conjugate to a given one with respect to system U is called an integral of ramification of system U *).

The relation of conjugation is reflexive, transitive and symmetric. Points lying on the same integral are conjugate but not necessarily vice versa. It is not difficult to ascertain that every function continuous in D and constant along the integrals of system U is also constant along every integral of ramification of system U. Thus, in order to construct the system in question, it suffices to define a system of form (1) having an integral of ramification everywhere-dense in E_3.

2. For a given positive number r it is not difficult to construct an example of an equation of the form

$$\frac{dy}{dx} = K_{(r)}(x,y), \tag{2}$$

where the function $K_{(r)}(x,y)$ is of class C^∞ in the zone $L: 0 \leqslant x \leqslant 1$, and such that

$$K_{(r)}(x,y) = 0 \tag{3}$$

outside the set Z,

$$\tfrac{1}{3} \leqslant x \leqslant \tfrac{2}{3} \quad \text{or} \quad \tfrac{3}{5} \leqslant x \leqslant \tfrac{4}{5}, \ -1 \leqslant y,$$

(4) *the integrals of equation* (2), $y = g(x;u,v)$, *passing trough an arbitrary point* (u,v) *of zone* L, *are defined in the interval* $[0,1]$,

$$g(0;u,v) = g(1;u,v) \quad \text{for} \quad (u,v) \in L, \tag{5}$$

$$g(\tfrac{1}{2};0,0) = r. \tag{6}$$

(Equation (2) together with the equation $dy/dx = 0$ will play an important rôle in our construction).

In the first stage we shall show that for an arbitrary number a and an arbitrary point $Y^* = (y_1^*, y_2^*)$ there exists a system $U(a, Y^*)$ of form (1) with the right sides $Q_i(x, y_1, y_2; a, Y^*)$ $(i = 1,2)$, defined in the layer $W(a)$: $a \leqslant x \leqslant a+1$, having the following properties:

(7) *functions* $Q_i(x, y_1, y_2; a, Y^*)$ *are of class* C^∞ *in the set* $W(a)$,

(8) *if a point* $P(x, y_1, y_2)$ *belongs to the set* $W(a) - H^*$ *where* H^* *is the plane passing through the point* $P^* = (0, y_1^*, y_2^*)$ *and the x-axis, then the integral* $Y = Y(x;P)$ *of system* $U(a, Y^*)$ *passing through the point* P *exists in the interval* $[a, a+1]$,

(9) $Y(a;P) = Y(a+1;P)$ for $P \in W(a) - H^*$,

(10) *point* (a, y_1^*, y_2^*) *is conjugate to the point* $(a,0,0)$,

(11) *functions* $Q_i(x, y_1, y_2; a, Y^*)$ $(i = 1,2)$ *vanish with all their derivatives on the planes* $x = a$ *and* $x = a+1$.

Relation (8) and the theorem on the continuous dependence of the integral upon the initial conditions imply the following property:

*) The notion of an integral of ramification has been introduced by T. Ważewski.

(12) *The sum of the integrals passing through the points of a set everywhere-dense on the plane $x=a$ is everywhere-dense in the layer $W(a)$.*

Besides, from relation (9) it follows (as can easily be verified) that

(13) *the points (a, Y), $(a+1, Y)$ are conjugate for an arbitrary point Y.*

In the proof we can restrict ourselves to the case $a=0$, $y_1^*>0$, $y_2^*=0$, because the general case can be reduced to this one for $|Y^*|>0$ *) by a rotation of the co-ordinate system around the x-axis and a translation parallel to the x-axis. Besides, we shall construct a system satisfying relations (7)-(11) not in the layer $W(0)$ but in the set V, formed from the layer $W(0)$ by the removal of the set $S(\frac{1}{5}\leqslant x\leqslant\frac{2}{5}$ or $\frac{3}{5}\leqslant x\leqslant\frac{4}{5}$, $y_1\geqslant -1$, $y_2=0)$.

It is easy to observe that the relation of conjugation is an invariant of a regular transformation. Therefore a suitable transformation of the set V upon the layer $W(0)$ transforms the above-mentioned system, defined in the set V, into the system $U(0, y_1^*, 0)$. It is also easy to ascertain that the formulas $x=\tilde{x}$, $y_1=A(x,\tilde{y}_2)\tilde{y}_1+(1-A(x,\tilde{y}_2))(-1-e^{-\tilde{y}_1})$, $y_2=\tilde{y}_2$ where the function $A(x,y)$ is of class C^∞ for $0\leqslant x\leqslant 1$ and an arbitrary y, $A(x,y)=0$ in the set J $\frac{1}{5}\leqslant x\leqslant\frac{2}{5}$ or $\frac{3}{5}\leqslant x\leqslant\frac{4}{5}$, $y=0$, $0<A(x,y)\leqslant 1$ outside the set J, $A(x,y)=1$ for $0\leqslant x\leqslant\frac{1}{10}$ and for $\frac{9}{10}\leqslant x\leqslant 1$, define the required transformation of the set V upon the layer $0\leqslant\tilde{x}\leqslant 1$.

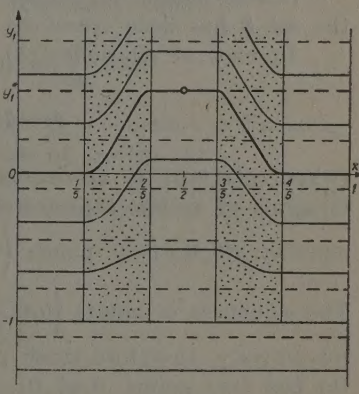

Fig. 1

We proceed now to define a system of form (1) in set V. (This system is shown in Fig. 1. The graphs of the integrals in the planes $y_2=c<0$ are given by discontinuous lines and in the planes $y_2=\gamma>0$ by continuous curves. The set S, lying in the plane $y_2=0$ and not belonging to the set V, is shown by dotted lines).

We set

(14) $Q_1(x, y_1, y_2)=K_{(y_1^*)}(x, y_1)$ *for* $y_2>0$,

(15) $Q_1(x, y_1, y_2)=0$ *for* $y_2<0$ *and in the plane* $y_2=0$ *outside the set* S *(in which we do not define function* Q_1 *at all).*

(16) $Q_2(x, y_1, y_2)=0$ *in the whole set* V.

In virtue of (3) function $Q_1(x, y_1 y_2)$ is of class C^∞, therefore, in view of (16) functions $Q_i(x, y_1 y_2)$ have property (7) in V. It is not difficult to

*) The case $y_1^*=0, y_2^*=0$ requires no explanation.

ascertain, taking into account relations (3)-(5) and (14)-(16), that functions $Q_i(x, y_1, y_2)$ and the system of form (1) defined with their aid satisfy properties (8), (9), (11).

In view of relations (15), (16) the straight lines

$$y_1 = y_1^*$$
$$y_2 = -\frac{1}{n}$$

are integrals of system (1) for $0 \leqslant x \leqslant 1$, i. e. points $\left(0, y_1^*, -\frac{1}{n}\right)$, $\left(\frac{1}{2}, y_1^*, -\frac{1}{n}\right)$ lie on the same integrals. It follows that points $(0, y_1^*, 0)$, $(\frac{1}{2}, y_1^*, 0)$, are directly conjugate, since the sequences of points $\left(0, y_1^*, -\frac{1}{n}\right)$, $\left(\frac{1}{2}, y_1^*, -\frac{1}{n}\right)$ tend to points $(0, y_1^*, 0)$, $(\frac{1}{2}, y_1^*, 0)$ respectively.

(These points do not lie on the same integral because from $W(0)$ the set S has been removed).

In view of relations (14), (16), (4), the curves

$$y_1 = g(x; 0, 0), \qquad y_2 = \frac{1}{n},$$

(where $g(x; 0, 0)$ is an integral of the equation $dy/dx = K_{O_1^*}(x, y)$) are integrals of system (1) for $0 \leqslant x \leqslant 1$, and thus in virtue of (6) and the equality $g(0; 0, 0) = 0$ points $\left(0, 0, \frac{1}{n}\right)$, $\left(\frac{1}{2}, y_1^*, \frac{1}{n}\right)$ lie on the same integrals. The sequences of points $\left(0, 0, \frac{1}{n}\right)$, $\left(\frac{1}{2}, y_1^*, \frac{1}{n}\right)$ tend to points $(0, 0, 0)$, $(\frac{1}{2}, y_1^*, 0)$ respectively, therefore these points are directly conjugate. In virtue of the fact that points $(0, y_1^*, 0)$ $(\frac{1}{2}, y_1^*, 0)$ are also directly conjugate, points $(0, 0, 0)$ and $(0, y_1^*, 0)$ are conjugate, i. e. relation (10) is satisfied, and thus the proof of the existence of system $U(a, Y^*)$ satisfying properties (7)-(11) in the layer $W(a)$ is completed.

3. Now let $Y^0, Y^1, Y^2 \ldots$ be a sequence of points everywhere-dense in the plane.

We shall define the required system in the following way:

$$\begin{aligned} Q_1(x, y_1, y_2) &= 0 \\ Q_2(x, y_1, y_2) &= 0 \end{aligned} \qquad \text{for} \quad x < 0 \tag{17}$$

$$\begin{aligned} Q_1(x, y_1, y_2) &= Q_1(x, y_1, y_2; i, Y^i) \\ Q_2(x, y_1, y_2) &= Q_2(x, y_1, y_2; i, Y^i) \end{aligned} \qquad \text{for} \quad i \leqslant x < i+1 \ \ (i = 0, 1, 2, \ldots). \tag{18}$$

In view of relations (7) (11) functions Q_i are of class C^∞ in the whole space. From relation (13) (and the transitivity of the relation of conjugation) it follows that for $i = 0, 1, 2, \ldots$ points $(i, 0, 0)$ are conjugate to point $(0, 0, 0)$. As a result of relations (18), (10) points (j, Y^j) are con-

jugate to points $(j,0,0)$ $(j=0,1,...)$ and thus also to point $(0,0,0)$, and since by property (13) points (i, Y^j) are for $i=0,1,2,...$ conjugate to point (j, Y^j) $(j=0,1,2,...)$, therefore

(19) *all the points* (i, Y^j) $(i,j=0,1,...)$ *are conjugate to point* $(0,0,0)$.

From definition (17) it follows that for $x \leqslant 0$ points (x, Y^j) are conjugate to points $(0, Y^j)$ since they lie on the same integrals, and thus also to point $(0,0,0)$, i. e. the set of points conjugate to point $(0,0,0)$ is everywhere-dense in the left half-space, and since from relations (18), (19), (12) it follows that points conjugate to point $(0,0,0)$ are every where dense in every layer $i \leqslant x \leqslant i+1$ $(i=0,1,2,...)$, therefore the integral of ramification of the system we have constructed, (to which point $(0,0,0)$ belongs), is everywhere-dense in the space. Thus the system constructed has the required properties

Remark. Every solution of a partial differential equation

$$\frac{dz}{dx} + Q_1(x, y_1, y_2)\frac{dz}{dy_1} + Q_2(x, y_1, y_2)\frac{dz}{dy_2} = 0$$

having a total differential in the whole space E_3 is constant in E_3 since, as we know, it is constant along the integrals of system (1).

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MATHEMATICS

A theorem on monotone mappings

by
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Presented by K. BORSUK on January 3, 1955

Only metric separable spaces will be considered. The mapping $f: X \to Y$ will be called *interior* if it maps open sets onto open ones; it will be called *strictly monotone* if there exists a positive number l such that for every $y \in f(X)$ the set $f^{-1}(y)$ is a continuum of diameter $\geqslant l$. We shall use the following abbreviations:
$\delta[A]$ — diameter of the set A; $K(A, \varepsilon) = \underset{x}{E}[\varrho(x, A) < \varepsilon]$, provided $\varepsilon > 0$.

THEOREM 1. *Let f: $X \to Y$ be a continuous interior strictly monotone mapping of a compact space X into a finite-dimensional space Y.*

Given $\varepsilon > 0$, there exists a finite collection of closure-disjoints, open subsets $G_1, \ldots, G_k$ of X such that $\delta[G_i] < \varepsilon$ for $1 \leqslant i \leqslant k$, and $f(\sum_{i=1}^{k} G_i) = f(X)$. *).

Theorem 1 can also be formulated as follows:

THEOREM 1′. *Let $f^{-1}(y)$, $y \in Y$, be a continuous decomposition* (cf. [1], § 39, VI) *of a compact space X in continua each of diameter greater than $l > 0$. Suppose that the space Y (the hyperspace of this decomposition) is of finite dimension. Let ε be a positive number.*

Then there exists a finite collection of open, closure-disjoints, subsets $G_1, \ldots, G_k$ of X such that $\delta[G_i] < \varepsilon$ and $f^{-1}(y) \cdot \sum_{i=1}^{k} G_i \neq 0$ for every $y \in Y$.

To establish the equivalence between 1 and 1′ it suffices to remark that the mapping of the space X onto the hyperspace of its upper semi-continuous decomposition is interior if and only if this decomposition is continuous (cf. [1], § 39, VI, 2).

Now, we shall prove some lemmas.

Suppose $\mathfrak{U} = \{U_1, \ldots, U_k\}$ is a finite covering of a space. Let λ_i be the number of indices j for which $\overline{U}_i \cdot \overline{U}_j \neq 0$ and let $\mu_i = \delta[U_i]$. Denote:

$$\lambda(\mathfrak{U}) = \max_{1 \leqslant i \leqslant k} \lambda_i \quad \text{and} \quad \mu(\mathfrak{U}) = \max_{1 \leqslant i \leqslant k} \mu_i.$$

*) This theorem was suggested by Professor Knaster.

LEMMA 1. *Suppose Y is a compact space of dimension m. Then there exists a natural number M and a sequence $\{\mathfrak{U}^n\}$ of finite open coverings of Y such that $\mu(\mathfrak{U}^n)\to 0$ and $\lambda(\mathfrak{U}^n)<M$.*

Proof. Let h be a homeomorphism mapping Y into the $(2m+1)$-dimensional cell Q. For the cell Q the lemma is obviously true, hence there exists a number M and a sequence $\{\mathfrak{V}^n\}$, $\mathfrak{V}^n=\{V_1^n,\dots,V_{k_n}^n\}$ of finite open coverings of Q such that $\mu(\mathfrak{V}^n)\to 0$ and $\lambda(\mathfrak{V}^n)<M$. We assert that the number M and the sequence $\{\mathfrak{U}^n\}$,

$$\mathfrak{U}^n=\{h^{-1}[h(Y)\cdot V_1^n],\dots,h^{-1}[h(Y)\cdot V_{k_n}^n]\},$$

have the desired properties.

Since h is a homeomorphism, the family $\mathfrak{U}^n$ forms a finite open covering of Y. For the same reason $\lambda(\mathfrak{U}^n)<M$. By the uniform continuity of h^{-1}, $\mu(\mathfrak{U}^n)\to 0$ and the proof is complete.

LEMMA 2. *Suppose there are given in a space B: a continuum A of diameter $\geqslant l>0$ and a collection of M sets $G_1,\dots,G_M$ such that $\delta[G_i]\leqslant\varepsilon$ for $i\leqslant M$. Let $f(x)=\varrho(x,\sum_{i=1}^{M}G_i)$.*

Then

$$\sup_{x\in A} f(x)\geqslant\frac{l-M\varepsilon}{2M}$$

Proof. Denote

$$A_i=E_{x\in A}\Big[\varrho\Big(x,\sum_{i=1}^{M}G_i\Big)=\varrho(x,G_i)\Big].$$

Let $a,b\in A$ and $\varrho(a,b)\geqslant l$. Since A is connected and the sets A_i form a closed covering A, there exists a subcollection $A_{k_1},\dots,A_{k_n}$ such that $a\in A_{k_1}$, $b\in A_{k_n}$ and $A_{k_{i-1}}\cdot A_{k_i}\neq 0$ for $i=2,\dots,n$. Let $a_1=a, a_{n+1}=b$ and $a_i\in A_{k_{i-1}}\cdot A_{k_i}$, $i=2,\dots,n$. Then

$$l\leqslant\varrho(a,b)\leqslant\varrho(a_1,a_2)+\varrho(a_2,a_3)+\dots+\varrho(a_n,a_{n+1})\leqslant$$
$$\leqslant\delta[A_{k_1}]+\delta[A_{k_2}]+\dots+\delta[A_{k_n}]\leqslant\sum_{i=1}^{M}\delta[A_i].$$

Hence at least one of the sets A_i is of diameter not smaller than l/M. Suppose this is the set A_1. Denote

$$\eta=\sup_{x\in A_1} f(x).$$

By definition of A_1,

$$\eta=\sup_{x\in A_1}\varrho(x,G_1)$$

and we infer that A_1 belongs to the closure of $K(G_1, \eta)$. But $\delta[K(G_1, \eta)] \leqslant \varepsilon + 2\eta$, hence

$$l/M \leqslant \varepsilon + 2\eta, \quad \text{i. e.} \quad \eta \geqslant \frac{l - M\varepsilon}{2M}.$$

Since $\sup_{x \in A} f(x) \geqslant \eta$ the lemma is proved.

LEMMA 3. *Let X, Y and f be as in theorem 1'. Let $\mathfrak{U}^n = \{U_1^n, \ldots, U_{k_n}^n\}$, $n = 1, 2, \ldots$, be a covering of Y. Denote*

$$\varrho_{n,k} = \sup_{x \in f^{-1}(U_k^n)} \sup_{y \in U_k^n} \varrho(x, f^{-1}(y)) \quad \text{and} \quad \varrho_n = \max_{1 \leqslant k \leqslant k_n} \varrho_{n,k}.$$

If $\mu(\mathfrak{U}^n) \to 0$, then also $\varrho_n \to 0$.

Proof. Suppose the lemma is false. Then there exists a positive number η, two sequences: $n_1 < n_2 < \ldots$ and $k_1 < k_2 < \ldots$ of natural numbers and two sequences of points: $\{x_m\}$ and $\{y_m\}$ such that

$$x_m \in f^{-1}(U_{k_m}^{n_m}), \quad y_m \in U_{k_m}^{n_m}; \tag{1}$$

$$\varrho(x_m, f^{-1}(y_m)) > \eta \quad \text{for each} \quad m. \tag{2}$$

Since X and Y are compact we may, if necessary, by extracting of subsequences, assume that

$$x_m \text{ converges to } x \text{ and } y_m \text{ converges to } y. \tag{3}$$

Since $\mu(\mathfrak{U}^n) \to 0$ it follows from (1) that $\varrho(f(x_m), y_m) \to 0$, thus

$$\lim f(x_m) = \lim y_m = y.$$

From the upper semicontinuity of the decomposition it follows then that $x \in f^{-1}(y)$. From the continuity it follows that there exists a sequence x'_m such that

$$x'_m \in f^{-1}(y_m); \tag{4}$$

$$\lim x'_m = x. \tag{5}$$

But (3) and (5) imply that $\varrho(x_m, x'_m) \to 0$; hence (4) implies $\varrho(x_m, f^{-1}(y_m)) \to 0$, and this contradicts (2).

Proof of Theorem 1'. Let M and $\{\mathfrak{U}^n\}$ be the natural number and the sequence of coverings of the space Y constructed as in lemma 1. Let ϱ_n be the sequence of numbers for the sequence $\{\mathfrak{U}^n\}$ constructed as in lemma 3. Let ε be a given positive number, and η a positive number such that

$$\eta < \min\left(\varepsilon, \frac{l}{2M}\right). \tag{6}$$

Let n be a fixed natural number such that

$$\varrho_n < \eta/2. \tag{7}$$

By lemma 3 such a number exists. Let $U_1,\dots,U_k$ be the members of the (fixed) covering $\mathfrak{U}^n$. The collection $\{G_m\}$ will be constructed stepwise.

Let x be an arbitrary point of $f^{-1}(U_1)$ and let $G_1=f^{-1}(U_1)\cdot K(x,\eta/2)$. Thus G_1 is an open subset of X of diameter not greater than η and $f(G_1)=U_1$. Suppose we have constructed a sequence $G_1,\dots,G_m$, where $m<k$, of open subsets of X such that

$$\bar{G}_i\cdot\bar{G}_j=0 \quad \text{if} \quad i\neq j; \tag{8}$$

$$f(G_i)=U_i; \tag{9}$$

$$\delta[G_i]\leqslant\eta. \tag{10}$$

We have to construct the set G_{m+1} such that the collection $G_1,\dots,G_{m+1}$ satisfies (8)-(10).

Let $A=f^{-1}(y)$, where y is an arbitrary point of U_{m+1}. We observe that by (9) no more than M of the sets $\bar{G}_1,\dots,\bar{G}_m$ intersect the closure of $f^{-1}(U_{m+1})$. Denote the sum of closures of these sets by G. Lemma 2 applied to the set $B=\overline{f^{-1}(U_{m+1})}$ yields a point x in A such that $\varrho(x,G)\geqslant \frac{l-M\eta}{2M}$, hence, by (6), $\varrho(x,G)>\eta/2$. Putting

$$G_{m+1}=f^{-1}(U_{m+1})\cdot K(x,\eta/2),$$

we infer that $\bar{G}_{m+1}\cdot G=0$. But from $G_{m+1}\subset f^{-1}(U_{m+1})$ it follows that $\bar{G}_{m+1}\cdot\sum_{i\leqslant m}\bar{G}_i\subset G$, therefore $\bar{G}_{m+1}\cdot\sum_{i\leqslant m}\bar{G}_i=0$ and the collection $G_1,\dots,G_{m+1}$ satisfies (8). Applying (7) we infer from the definition of ϱ_n that $f(G_{m+1})=U_{m+1}$, hence (9) is satisfied also. From $\delta[K(x,\eta/2)]\leqslant\eta$ and $G_{m+1}\subset K(x,\eta/2)$ follows (10).

Therefore in a finite number of steps we can construct a collection $G_1,\dots,G_k$ of open, closure-disjoints, subsets of X, each set of diameter not greater than $\eta<\varepsilon$, and such that

$$f\Big(\sum_{i=1}^{k}G_i\Big)=\sum_{i=1}^{k}U_i=Y.$$

The proof of Theorem 1′, therefore also of Theorem 1, is thus complete.

Returning to the terminology of Theorem 1 we set two problems:

Problem 1. Is the hypothesis that $f(X)$ is finite-dimensional essential?

Problem 2. Is the hypothesis that f is interior essential? It can be proved that the hypothesis that f is strictly monotone is essential.

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MATHEMATICS

On arbitrary systems of linear equations over semi-simple rings

by

A. KERTÉSZ

Communicated by K. KURATOWSKI on December 29, 1954

In this paper we shall always denote by a semi-simple ring a ring containing no nilpotent left ideal $\neq 0$ and satisfying the descending chain condition for left ideals. According to the known Wedderburn-Artin structure theorem, such a ring is isomorphic to a direct sum of a finite number of rings, each of which is isomorphic to the complete ring of linear transformations in a suitable finite-dimensional vector space over a skew field.

Let R be an arbitrary ring. Consider a system of indeterminates x_α, $\alpha \in A$, and, moreover, a system of linear forms f_β, $\beta \in B$ over R:

$$f_\beta = a_{\beta 1} x_{\alpha_1} + \ldots + a_{\beta k} x_{\alpha_k} \qquad (a_{\beta j} \in R),$$

each of which contains an arbitrary but finite number of x_α's; A and B being arbitrary (finite or infinite) sets of indices. If b_β are given elements of R, then the left-handed system

$$f_\beta = b_\beta \qquad (\beta \in B) \tag{1}$$

of linear equations over R is called *compatible* if every left-handed linear dependence relation over R for an arbitrary finite number of the linear forms f_β is also satisfied by the corresponding b_β's. We seek all solutions $x_\alpha = r_\alpha \in R$ of this system and we say that the ring R admits *the classical theory of linear equations* provided that for every compatible system (1) of linear equations over R all solutions in R can be obtained by a system of solving formulae of the form

$$x_\alpha = c_\alpha + \sum_{j=1}^{m_\alpha} d_{\alpha j} t_{\alpha_j} \qquad (c_\alpha, d_{\alpha j} \in R),$$

where c_α is a (finite) left linear combination *) of some b_β's over R, $\{t_\gamma\}$

*) It is useful to recall the classical treatment of the solution of a finite system of linear equations over a skew field.

($\gamma \in \Gamma$) is a set of parameters with values freely chosen from R, and m_α is a certain natural number.

A new characterization of semi-simple rings is contained in the following result: *A ring admits the classical theory of linear equations if, and only if, it is semi-simple.*

In detail:

THEOREM 1. *A ring R is semi-simple if, and only if, every compatible system* (1) *of linear equations over R possesses a solution $x_\alpha = c_\alpha \in R$ such that each c_α is a left linear combination of some b_β's.*

THEOREM 2. *If R is a semi-simple ring, then all solutions of an arbitrary homogeneous* *) *system of linear equations over R can be obtained by a system of left linear combinations (over R) of a set of parameters with values freely chosen from R.*

As immediate consequences we have:

COROLLARY 1. *An arbitrary system of linear equations over a semi-simple ring R admits a solution in R if, and only if, any finite subsystem has a solution in R.*

COROLLARY 2. *Every system of linear equations over a semi-simple ring contains a maximal solvable subsystem.*

The results above contain a generalization of some results by Gacsályi and Szele [1], [2].

Proofs will be given in [3].

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*) i. e. every $b_\beta = 0$ in (1).

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MATHÉMATIQUE

Un théorème sur les espaces complets et ses applications à l'étude de la connexité locale

par

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1. THÉORÈME 1. *Soient $\mathfrak{X}$ un espace métrique et $f_1, f_2, \ldots$ une suite de fonctions continues, définies sur $\mathfrak{X}$ et à valeurs réelles. Admettons que toute suite de points $x_1, x_2, \ldots$ de $\mathfrak{X}$ qui satisfait aux deux conditions suivantes est convergente:*

(1) *condition de Cauchy* *),

(2) *la suite $f_m(x_1), f_m(x_2), \ldots$ est bornée, quel que soit m.*

Dans ces hypothèses, $\mathfrak{X}$ est homéomorphe à un espace complet.

D'une façon plus précise: *en désignant par $|x-y|$ la distance dans l'espace $\mathfrak{X}$ et en posant*

$$\|x-y\| = |x-y| + \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{|f_m(x)-f_m(y)|}{1+|f_m(x)-f_m(y)|}, \tag{3}$$

on définit une „nouvelle" distance dans $\mathfrak{X}$ qui est complète et qui équivaut topologiquement à la distance primitive, c'est-à-dire que l'on a l'équivalence

$$\{\lim_{i=\infty} \|x_i - x\| = 0\} \equiv \{\lim_{i=\infty} |x_i - x| = 0\}. \tag{4}$$

Démonstration. Pour se convaincre que $\|x-y\|$ est une distance. posons $f(x) = [f_1(x), f_2(x), \ldots]$.

f est donc une transformation continue de l'espace $\mathfrak{X}$ en sous-ensemble de l'espace $\mathcal{E}^{\aleph_0}$ de Fréchet (espace des suites infinies des nombres réels) et $\|x-y\|$ est la distance du point $[x, f(x)]$ au point $[y, f(y)]$, la distance de deux points $\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \ldots)$ et $\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2, \ldots)$ de $\mathcal{E}^{\aleph_0}$ étant définie comme d'habitude (cf. [1], p. 82, ou bien [2], § 15, V, 2) par l'égalité

*) On dit qu'une suite de points $x_1, x_2, \ldots$ *satisfait à la condition de Cauchy* lorsqu'à tout $\varepsilon > 0$ correspond un k tel que pour $i > k$ on a $|x_i - x_k| < \varepsilon$. Si toute suite satisfaisant à la condition de Cauchy est convergente, l'espace est dit *complet*.

$$|\mathfrak{z}-\mathfrak{y}|=\sum_{m=1}^{\infty}\frac{1}{2^m}\cdot\frac{|\mathfrak{z}_m-\mathfrak{y}_m|}{1+|\mathfrak{z}_m-\mathfrak{y}_m|},$$

et le produit cartésien $\mathfrak{X}\times\mathcal{E}^{\aleph_0}$ étant métrisé par la formule

$$|(x,\mathfrak{z})-(y,\mathfrak{y})|=|x-y|+|\mathfrak{z}-\mathfrak{y}|.$$

La fonction f étant continue, la correspondance qui attache au point x de $\mathfrak{X}$ le point $[x,f(x)]$ de la „courbe“ $\mathfrak{z}=f(x)$, est une homéomorphie (cf. [2], § 24, XI, 2).

L'équivalence (4) se trouve ainsi établie.

Nous allons montrer à présent que l'espace $\mathfrak{X}$ métrisé par la distance $\|x-y\|$ est complet.

Admettons à ce but que la suite $x_1,x_2,\ldots$ satisfait à la condition de Cauchy par rapport à la distance $\|x-y\|$. Comme $|x-y|\leqslant\|x-y\|$, cette suite satisfait donc à la condition de Cauchy par rapport à la distance $|x-y|$. Tout revient donc à établir la condition (2); car, dès qu'elle sera établie, on en conclura que la suite $x_1,x_2,\ldots$ converge vers un point x de $\mathfrak{X}$, c'est-à-dire que

$$\lim_{i=\infty}|x_i-x|=0, \quad \text{d'où} \quad \lim_{i=\infty}\|x_i-x\|=0,$$

en vertu de (4).

Or, supposons par impossible que la condition (2) soit en défaut, c'est-à-dire qu'il existe un indice m tel que la suite $f_m(x_1),f_m(x_2),\ldots$ soit non bornée.

Il existe donc une suite d'indices $j_1<j_2<\ldots$ telle que

$$\lim_{i=\infty}|f_m(x_{j_i})|=\infty. \tag{5}$$

La suite $x_{j_1},x_{j_2},\ldots$ satisfaisant à la condition de Cauchy (par rapport à la distance $\|x-y\|$), il existe un indice k tel que

$$\|x_{j_i}-x_{j_k}\|<\frac{1}{2^{m+1}} \quad \text{pour} \quad i>k,$$

d'où, selon (3),

$$\frac{1}{2^m}\cdot\frac{|f_m(x_{j_i})-f_m(x_{j_k})|}{1+|f_m(x_{j_i})-f_m(x_{j_k})|}\leqslant\|x_{j_i}-x_{j_k}\|<\frac{1}{2^{m+1}},$$

c'est-à-dire

$$\frac{1}{1+\dfrac{1}{|f_m(x_{j_i})-f_m(x_{j_k})|}}<\frac{1}{2}. \tag{6}$$

En faisant tendre i vers ∞ et en tenant compte de la formule (5), on parvient à une contradiction, puisque le membre gauche de l'inégalité (6) tend vers 1.

2. Remarque. Un cas particulier du théorème 1 présente le théorème bien connu de P. Alexandroff, d'après lequel *tout sous-ensemble G_δ d'un espace complet est homéomorphe à un espace complet* (voir [4] et [5]).

Soit, en effet, $\mathfrak{X}$ un sous-ensemble G_δ d'un espace complet $\mathfrak{Y}$. On a donc $\mathfrak{Y}-\mathfrak{X}=F_1+F_2+\ldots$, somme d'ensembles fermés. En posant $f_m(x)=1:\varrho(x,F_m)$ pour $x\in\mathfrak{X}$, on constate facilement que les hypothèses du théorème 1 sont réalisées (cf. [2], § 29, VI), donc que $\mathfrak{X}$ est homéomorphe à un espace complet.

De plus, la formule (3) permet de remétriser l'espace $\mathfrak{X}$ de façon qu'il devienne complet.

3. Applications à la connexité locale homologique. On dit qu'un espace compact F est lc^n (localement connexe homologiquement en dimensions $\leqslant n$), lorsqu'à tout $t>0$ correspond un $\eta>0$ tel que tout cycle (de Vietoris) de dimension $\leqslant n$ et de diamètre $<\eta$ est homologue à 0 sur un ensemble de diamètre $\leqslant t$. La borne supérieure des nombres $\eta(\leqslant t)$ de ce genre est désignée par $\delta_F^{(n)}(t)$ ([6], 441).

Évidemment:

(7) *la fonction $\delta_F^{(n)}(t)$ est non-décroissante et positive pour $t>0$ (si F est lc^n).*

$\mathfrak{Y}$ étant un espace compact, désignons par $lc^n(\mathfrak{Y})$ l'espace de tous ses sous-ensembles lc^n fermés, en entendant par suite convergente vers F toute suite $F_1,F_2,\ldots$ (d'éléments de $lc^n(\mathfrak{Y})$) satisfaisant aux deux conditions suivantes:

$$\lim_{i=\infty}\operatorname{dist}(F_i,F)=0 \text{ *)}, \tag{8}$$

$$\liminf_{i=\infty}\delta_{F_i}(t)>0 \quad \text{pour tout} \quad t>0 \text{ **)}. \tag{9}$$

Comme l'ont montré P. A. White [8] et — plus directement E. G. Begle [6], l'espace $lc^n(\mathfrak{Y})$ est métrisable. Begle a défini en effet la distance entre deux éléments F et H de cet espace par la formule:

$$\varrho_n(F,H)=\operatorname{dist}(F,H)+\int_0^1|\delta_F^{(n)}(t)-\delta_H^{(n)}(t)|\,dt \text{ ***)}. \tag{10}$$

Begle a démontré en outre que l'espace $lc^n(\mathfrak{Y})$ est homéomorphe à un espace complet, sans toutefois donner une définition d'une métrique complète (la métrique (10) n'est pas complète).

Or, le théorème 1 permet de définir explicitement une distance dans l'espace $lc^n(\mathfrak{Y})$ par rapport à laquelle cet espace est complet.

*) dist désigne la distance de Hausdorff.

**) Pour les renseignements bibliographiques concernant cette notion de convergence (dite régulière), voir [7].

***) La formule précitée diffère de celle de Begle de façon inessentielle.

Voici cette définition:

$$\bar{\varrho}_n(F,H)=\varrho_n(F,H)+\sum_{m=1}^{\infty}\frac{1}{2^m}\cdot\frac{|f_m(F)-f_m(H)|}{1+|f_m(F)-f_m(H)|} \tag{11}$$

$$=\varrho_n(F,H)+\sum_{m=1}^{\infty}\frac{1}{2^m}\cdot\frac{|g_m(F)-g_m(H)|}{|g_m(F)-g_m(H)|+g_m(F)\cdot g_m(H)},$$

où

$$g_m(F)=\int_0^{1/m}\delta_F^{(n)}(t)\,dt \quad \text{et} \quad f_m(F)=\frac{1}{g_m(F)}. \tag{12}$$

Il s'agit de démontrer que les hypothèses du théorème 1 sont vérifiées (en substituant à $\mathfrak{X}$ l'espace $lc^n(\mathcal{Y})$ et en entendant par distance dans cet espace la distance $\varrho_n(F,H)$).

Remarquons d'abord que $g_m(F)$, donc $f_m(F)$, est une fonctionnelle continue de la variable F, puisque

$$|g_m(F)-g_m(H)|=\left|\int_0^{1/m}[\delta_F^{(n)}(t)-\delta_H^{(n)}(t)]\,dt\right|$$

$$\leqslant\int_0^{1/m}|\delta_F^{(n)}(t)-\delta_H^{(n)}(t)|\,dt\leqslant\varrho_n(F,H).$$

En outre (d'après (7)), $g_m(F)>0$ pour tout $F\in lc^n(\mathcal{Y})$.

Puis, la condition que la suite $f_m(F_1),f_m(F_2),\ldots$ est bornée, c'est-à-dire la condition

$$\liminf_{i=\infty}g_m(F_i)>0 \quad \text{pour} \quad m=1,2,\ldots, \tag{13}$$

implique la condition (9).

Admettons, en effet, que

$$\liminf_{i=\infty}\delta_{F_i}^{(n)}(t_0)=0.$$

Il existe donc une suite $j_1<j_2<\ldots$ telle que

$$\lim_{i=\infty}\delta_{F_{j_i}}^{(n)}(t_0)=0, \quad \text{d'où} \quad \lim_{i=\infty}\delta_{F_{j_i}}^{(n)}(t)=0 \quad \text{pour} \quad t<t_0,$$

d'après (7).

Soit m un entier tel que $1/m<t_0$. Il vient

$$\lim_{i=\infty}\int_0^{1/m}\delta_{F_{j_i}}^{(n)}(t)=0, \quad \text{c.-à-d.} \quad \lim_{i=\infty}g_m(F_{j_i})=0, \quad \text{d'où} \quad \liminf_{i=\infty}g_m(F_i)=0.$$

Ceci établi, tout se réduit à démontrer que toute suite $F_1,F_2,\ldots$ d'éléments de l'espace $lc^n(\mathcal{Y})$ qui satisfait à la condition de Cauchy par rapport à la distance ϱ_n, ainsi qu'à la condition (13), converge vers un

élément F de $lc^n(\mathcal{Y})$, la convergence signifiant — comme nous l'avons dit auparavant — que les conditions (8) et (9) sont vérifiées.

Or, l'inégalité $\operatorname{dist}(F,H) \leqslant \varrho_n(F,H)$ (cf. (10)) implique aussitôt que toute suite qui satisfait à la condition de Cauchy par rapport à la distance ϱ_n, lui satisfait, à plus forte raison, par rapport à la distance de Hausdorff. L'espace de tous les sous-ensembles fermés de l'espace (compact) $\mathcal{Y}$ étant complet, il existe un F fermé satisfaisant à la condition (8). Or, comme l'a démontré Begle ([6], lemme 4) les conditions (8) et (9) (donc (8) et (13)) impliquent que $F \in lc^n(\mathcal{Y})$.

Il est ainsi établi que les hypothèses du théorème 1 sont vérifiées. On en conclut, en rapprochant les égalités (3) et (11), que l'espace $lc^n(\mathcal{Y})$ est complet relativement à la distance $\bar{\varrho}_n$.

4. Applications à la connexité locale homotopique. Rappelons qu'un espace compact F est dit LC^n (localement connexe homotopiquement en dimensions $\leqslant n$), lorsqu'à tout $t>0$ correspond un $\eta>0$ tel que toute transformation continue de la sphère $\mathcal{S}_k$ (où $k \leqslant n$) en sous-ensemble de F de diamètre $<\eta$ se laisse étendre à une transformation continue de la boule $\mathcal{Q}_{k+1}$ *) en sous-ensemble de F de diamètre $\leqslant t$. Désignons par $\alpha_F^{(n)}(t)$ la borne supérieure des nombres $\eta(\leqslant t)$ et par $LC^n(\mathcal{Y})$ la famille des sous-ensembles fermés LC^n de l'espace compact $\mathcal{Y}$; la convergence dans l'espace $LC^n(\mathcal{Y})$ est définie de façon tout à fait analogue à celle dans l'espace $lc^n(\mathcal{Y})$, sauf que l'on remplace δ par α (voir [9]).

On peut démontrer [10] que tous les énoncés du N° 3 restent valables, en remplaçant δ par α et lc par LC.

On est conduit ainsi à cette conclusion: *$\mathcal{Y}$ étant compact, l'espace $LC^n(\mathcal{Y})$ est complet, la distance entre deux éléments F et H de cet espace étant définie comme identique à*

$$\operatorname{dist}(F,H) + \int_0^1 |\alpha_F^{(n)}(t) - \alpha_H^{(n)}(t)|\,dt + \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{|h_m(F) - h_m(H)|}{|h_m(F) - h_m(H)| + h_m(F)\cdot h_m(H)} \tag{14}$$

où

$$h_m(F) = \int_0^{1/m} \alpha_F^{(n)}(t)\,dt.$$

Ajoutons que, dans l'hypothèse supplémentaire que $\mathcal{Y}$ est de dimension n, l'ensemble $LC^n(\mathcal{Y})$ coïncide avec celui des sous-ensembles fermés de $\mathcal{Y}$ qui sont des rétractes absolus de voisinage (cf. [3], p. 289). On retrouve ainsi le théorème de Borsuk ([11], p. 198): *l'espace des sous-ensembles fermés d'un espace compact de dimension finie qui sont des ré-*

*) Nous entendons par sphère $\mathcal{S}_k$ l'ensemble des points p de l'espace euclidien à $k+1$ dimensions tels que $|p|=1$, et par boule $\mathcal{Q}_{k+1}$ — celui des points p tels que $|p| \leqslant 1$.

tractes absolus de voisinage se laisse métriser de façon à devenir espace complet, la convergence dans cet espace étant entendue dans le sens précité (de Curtis). La distance définie ci-dessus fournit la métrisation demandée (elle est d'ailleurs bien différente de celle que Borsuk a défini).

5. Remarques finales. La définition de la distance (complète) dans les espaces $lc^n(\mathcal{Y})$ et $LC^n(\mathcal{Y})$ peut être simplifiée légèrement en omettant dans (11), resp. dans (14), le terme

$$\int_0^1 |c_F^{(n)}(t) - c_H^{(n)}(t)|\,dt \quad \text{resp.} \quad \int_0^1 |a_F^{(n)}(t) - a_H^{(n)}(t)|\,dt.$$

Cela se déduit facilement du théorème suivant qui présente une généralisation du théorème 1.

THÉORÈME 2. *Soient $\mathcal{X}$ et $\mathcal{Y}$ deux espaces, dont le premier est un espace L (de Fréchet) et le deuxième est métrique. Soit h une transformation biunivoque et continue de $\mathcal{X}$ en $\mathcal{Y}$. Soit $f_1, f_2, \ldots$ une suite de fonctions continues, définies sur $\mathcal{X}$ et à valeurs réelles. Admettons que toute suite $x_1, x_2, \ldots$, telle que la suite $h(x_1), h(x_2), \ldots$ satisfait à la condition de Cauchy et que la condition* (2) *est réalisée, soit convergente.*

Dans ces hypothèses, l'espace $\mathcal{X}$ devient complet en définissant la distance par la formule (3), *où $|x-y|$ est remplacé par $|h(x)-h(y)|$.*

La démonstration du théorème 2 est tout à fait analogue à celle du théorème 1.

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MATHÉMATIQUE

Sur la notion de diviseur et de multiple des transformations

par

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1. Étant donnés deux espaces compacts X_0 et Y_0, désignons par $Y_0^{X_0}$ l'espace — metrisé comme d'habitude — de toutes les fonctions continues transformant X_0 en sous-ensembles de Y_0. Nous allons supposer toujours que Y_0 est un ensemble ANR (= rétracte absolu de voisinage). La composante de $Y_0^{X_0}$, contenant la fonction f, sera désignée par (f) et appellée *classe de* f. Deux fonctions appartenant à la même classe sont dites *homotopes*.

Une fonction $f \in Y_0^{X_0}$ est dite *prolongeable* sur un espace $X \supset X_0$ lorsqu'il existe une fonction $f' \in Y_0^X$ telle qu'elle coïncide avec f dans X_0. Bien entendu, lorsque f est prolongeable sur X, chaque fonction g homotope à f est aussi prolongeable sur X. Les fonctions prolongeables sur chaque espace compact $X \supset X_0$ coïncident avec les fonctions homotopes aux constantes; elles vont être appellées *zéro-fonctions* et leurs classes — *zéro-classes*.

2. Une fonction $f \in Y_0^{X_0}$ sera dite *diviseur* d'une fonction $g \in Y_0^{X_0}$ (en symbole $f|g$) lorsque pour tout espace compact $X \supset X_0$ l'existence d'un prolongement $f' \in Y_0^X$ de f entraîne l'existence d'un prolongement $g' \in Y_0^X$ de g. On dira alors que g est *multiple* de f.

Évidemment, les relations $f|g$ et $g|h$ impliquent $f|h$. En outre on voit sans peine que les zéro-fonctions appartenant à $Y_0^{X_0}$ coïncident avec les fonctions qui sont multiples de chaque fonction $f \in Y_0^{X_0}$. Puisque les relations $f|g$, $f^* \in (f)$, $g^* \in (g)$ impliquent $f^*|g^*$, la relation $f|g$ est en réalité une relation entre les classes (f) et (g). Au lieu de dire que f est un diviseur de g, nous dirons aussi que la classe (f) est un *diviseur* de la classe (g).

3. Théorème. *Pour qu'une fonction $f \in Y_0^{X_0}$ soit un diviseur d'une fonction $g \in Y_0^{X_0}$, il faut et il suffit qu'il existe une fonction $\varphi \in Y_0^{Y_0}$ telle que g soit homotope à φf.*

Démonstration. Suffisance: Lorsque f admet un prolongement $f' \in Y_0^X$, la fonction $\psi f' \in Y_0^X$ constitue un prolongement de la fonction $\psi f \in \Upsilon_0^{X_0}$.

Nécessité *): Considérons le produit Cartésien $X_1 = X_0 \times \langle 0; 1\rangle$ de l'espace X_0 et de l'intervalle $0 \leqslant t \leqslant 1$. Sans compromettre la généralité, nous pouvons admettre que les espaces X_0, X_1 et Y_0 sont disjoints deux-à-deux. Identifions dans $X_0 + X_1 + Y_0$ tout point $(x,0) \in X_0 \times (0) \subset X_1$ avec le point $x \in X_0$ et tout point $(x,1) \in X_0 \times (1) \subset X_1$ avec le point $f(x) \in Y_0$. On obtient ainsi de $X_0 + X_1 + Y_0$ un espace compact X contenant X_0 et Y_0 **). En posant

$$f'(x,t) = f(x) \quad \text{pour tout} \quad x \in X_0 \quad \text{et} \quad 0 \leqslant t \leqslant 1,$$

$$f'(y) = y \quad \text{pour tout} \quad y \in Y_0,$$

on obtient une fonction $f' \in Y_0^X$ constituant un prolongement de la fonction f. Or, pour tout multiple g de f, il existe un prolongement $g' \in Y_0^X$ En posant

$$\psi(y) = g'(y) \quad \text{pour tout} \quad y \in Y_0,$$

on définit une fonction $\psi \in \Upsilon_0^{Y_0}$. En outre, en posant

$$g_t(x) = g'(x,t) \quad \text{pour tout} \quad x \in X_0 \quad \text{et} \quad 0 \leqslant t \leqslant 1,$$

on obtient une famille continue $\{g_t\}$ joignant dans $Y_0^{X_0}$ la fonction $g_0 = g$ avec la fonction g_1 définie par la formule

$$g_1(x) = g'(x,1) = g'[f(x)] = \psi f(x) \quad \text{pour tout} \quad x \in X_0.$$

Or, g est homotope à ψf, c. q. f. d.

4. Toute fonction $f \in \Upsilon_0^{X_0}$ établit une transformation homomorphe (un caractère) $\chi_f^{(\mathfrak{A},n)}$ du groupe n-dimensionnel d'homologie $H^n(X_0, \mathfrak{A})$ de X_0 (où $\mathfrak{A}$ désigne le groupe des coefficients) dans le groupe $H^n(Y_0, \mathfrak{A})$. Le théorème précédent implique le

Corollaire 1. *Lorsque $f, g \in \Upsilon_0^{X_0}$ et $f \mid g$, il existe un endomorphisme χ du groupe $H^n(Y_0, \mathfrak{A})$ tel que $\chi_g^{(\mathfrak{A},n)} = \chi(\chi_f^{(\mathfrak{A},n)})$.*

Dans le cas où Y_0 est une variété orientable de dimension n, l'endomorphisme χ du groupe $H^n(Y_0, \mathfrak{A})$, établi par une fonction $\psi \in Y_0^{Y_0}$, fait correspondre à tout élément $(\gamma) \in H^n(Y_0, \mathfrak{A})$ l'élément $m(\gamma)$, où m désigne le degré de la fonction ψ. Or, nous obtenons le

Corollaire 2. *Lorsque Y_0 est une variété orientable de dimension n et la fonction $f \in \Upsilon_0^{X_0}$ est un diviseur de la fonction $g \in \Upsilon_0^{X_0}$, il existe un*

*) La partie essentielle de cette démonstration coïncide avec le raisonnement suivi par R. H. Fox [2], p. 44, th. 3,1.

**) L'espace X constitue le "mapping cylinder" dans le sens de J. H. C. Whitehead [8].

nombre entier m tel, que pour tout vrai cycle γ de dimension n situé dans X_0 (avec coefficients arbitraires) on a

$$g(\gamma) \sim m \cdot f(\gamma) \quad \text{dans} \quad Y_0.$$

5. Deux fonctions $f, g \in Y_0^{X_0}$ sont dites *homologues* lorsque le caractère $\chi_f^{(\mathfrak{A},n)}$ coïncide avec le caractère $\chi_g^{(\mathfrak{A},n)}$ quel que soit le groupe de coefficients $\mathfrak{A}$ et $n = 0, 1, 2, \ldots$ Les fonctions homotopes sont toujours homologues. Dans le cas où $\dim X_0 \leqslant n$ et Y_0 est une sphère euclidienne de dimension n (ce cas sera cité comme le *cas de Hopf*) le réciproque est aussi vrai (voir [5] et [3]). Dans le cas de Hopf les classes d'homotopie constituent un groupe abélien (nommé le *groupe de Hopf*), dans lequel l'opération de l'addition fait correspondre aux classes $(f), (g) \in Y_0^{X_0}$ comme leur somme la classe $(h) \subset Y_0^{X_0}$ telle, que

$$\chi_h^{(\mathfrak{A},n)} = \chi_f^{(\mathfrak{A},n)} + \chi_g^{(\mathfrak{A},n)},$$

quel que soit le groupe $\mathfrak{A}$.

COROLLAIRE 3. *Dans le cas de Hopf la relation $f \backsim g$, où $f, g \in Y_0^{X_0}$ équivaut à l'existence d'un nombre entier m tel, que $(g) = m \cdot (f)$, où la multiplication de (f) par le nombre m est entendue dans le sens établi dans le groupe de Hopf de X_0.*

6. D'après un théorème de Freudenthal [4] le groupe de Hopf est isomorphe au groupe des caractères avec les valeurs appartenant au groupe $\mathfrak{R}_1$ des nombres réels réduits modulo 1, du groupe $H^n(X_0, \mathfrak{R}_1)$. Lorsque le groupe de Hopf de X_0 a un système fini de générateurs (ce qui a lieu en particulier lorsque X_0 est un polytope) alors le cas de Hopf sera dit *élémentaire*. Dans un groupe abélien ayant un système fini de générateurs, pour tout élément $a \neq 0$, il n'existe qu'un nombre fini des éléments b tels que $a = m \cdot b$ pour un m entier. En tenant compte du corollaire 3 on en déduit:

COROLLAIRE 4. *Dans le cas élémentaire de Hopf toute classe $(g) \subset Y_0^{X_0}$ qui n'est pas une zéro-classe n'a qu'un nombre fini de diviseurs.*

En particulier, pour un polytope X_0 de dimension $\leqslant n$ toute transformation continue de X_0 en une sphère n-dimensionnelle Y_0, non homotope à une constante, n'a qu'un nombre fini de diviseurs.

7. Nous allons dire que deux fonctions $f, g \in Y_0^{X_0}$ sont *conjuguées* lorsque f/g et g/f, c'est-à-dire lorsque pour tout espace compact $X \supset X_0$ la prolongeabilité de f sur X équivaut à la prolongeabilité de g sur X. Les classes (f) et (g) seront aussi dites *conjuguées*. Lorsque $\varphi \in Y_0^{Y_0}$ est une homéomorphie, alors pour toute fonction $f \in Y_0^{X_0}$ les fonctions f et φf sont conjuguées. La question reste ouverte, si inversement pour f et g conjuguées il existe toujours une homéomorphie $\varphi \in Y_0^{Y_0}$ telle que g et φf sont homotopes.

Il résulte du corollaire 4 que dans le cas élémentaire de Hopf pour toute fonction $f \in Y_0^{X_0}$ il n'existe qu'un nombre fini de fonctions dont les classes sont conjuguées avec (f). D'autre part, il est facile de donner des exemples des espaces X_0 et Y_0 tels qu'il existe des classes $(f) \subset Y_0^{X_0}$ pour lesquelles le nombre de classes conjuguées est infini.

8. Une fonction $g \in Y_0^{X_0}$ sera appellée *primitive*, lorsque tout diviseur de g est conjugué avec g, c'est-à-dire lorsque f/g implique g/f. La classe (g) d'une fonction primitive g sera dite *primitive*. On démontre sans peine que, lorsque $(f) \subset Y_0^{X_0}$ n'a qu'un nombre fini des diviseurs, il existe parmi eux des fonctions primitives.

COROLLAIRE 5. *Dans le cas élémentaire de Hopf il existe pour toute fonction $f \in Y_0^{X_0}$ un diviseur primitif.*

Il est facile d'observer, que, dans le cas où X_0 est le solenoide de van Dantzig (voir [1] et [6]) et Y_0 une circonférence, les fonctions primitives n'existent pas.

PROBLÈME 1. *Soient X_0 et Y_0 deux polytopes. Existe-il pour toute fonction $f \in Y_0^{X_0}$ un diviseur primitif?*

9. Le nombre des différents multiples d'une classe $(f) \subset Y_0^{X_0}$ sera dit *l'ordre de la classe* (f) ou *l'ordre de la fonction* f. Il est facile de donner des exemples des espaces X_0 et Y_0 (même appartenant au cas de Hopf) tels, que $Y_0^{X_0}$ contient des fonctions d'ordre arbitraire $\leqslant \aleph_0$. Dans le cas élémentaire de Hopf il résulte du corollaire 3, qu'il n'existe dans $Y_0^{X_0}$ qu'un nombre fini de classes dont les ordres sont finis.

PROBLÈME 2. *Soient X_0 et Y_0 deux polytopes. Est-il vrai que le nombre de classes $(f) \subset Y_0^{X_0}$, dont les ordres sont finis, est fini?*

10. Une fonction $f \in Y_0^{X_0}$ est dite *unitaire* lorsque pour tout $g \in Y_0^X$ on a f/g. Dans le cas où les espaces X_0 et Y_0 sont homéomorphes, une fonction $f \in Y_0^{X_0}$ transformant X_0 sur Y_0 topologiquement est toujours unitaire. En effet, lorsque f' est un prolongement de f sur $X \supset X_0$ alors pour tout $g \in Y_0^{X_0}$ il suffit de poser

$$g'(x) = gf^{-1}f'(x) \quad \text{pour tout} \quad x \in X,$$

afin d'obtenir un prolongement g' de g sur X.

Évidemment, toute fonction conjuguée avec une fonction unitaire est unitaire et toutes les fonctions unitaires sont conjuguées.

Dans le cas de Hopf on conclut du corollaire 3 que f est unitaire, si et seulement si la classe (f) constitue l'élément engendrant le groupe de Hopf de X_0. Or, dans ce cas le nombre des classes unitaires est fini.

L'exemple suivant montre que, même pour les transformations des polytopes, le nombre des classes unitaires peut être infini. Soit $X_0 = Y_0 = \mathfrak{R}_1 \times \mathfrak{R}_1$, où $\mathfrak{R}_1$ désigne le groupe (additif) des nombres réels réduits

modulo 1 avec la topologie usuelle. Or, $X_0 = Y_0$ est homéomorphe à la surface du tore. Considérons, pour tout m entier, la fonction $f \in Y_0^{X_0}$ définie par la formule

$$f_m(\xi, \eta) = (\xi + m\eta, \eta).$$

On voit aisément que f_m est une homéomorphie, donc une fonction unitaire. En outre $(f_m) \neq (f_{m'})$ pour $m \neq m'$.

11. Il est quelquefois commode de modifier la notion de diviseur d'une fonction et les notions dérivées de la manière suivante: Une fonction $f \in Y_0^{X_0}$ sera dite *diviseur* d'une fonction $g \in Y_0^{X_0}$ *en dimension* m (en symbole: $f \underset{m}{|} g$) lorsque pour tout espace compact $X \supset X_0$ satisfaisant à la condition $\dim(X - X_0) \leqslant m$, l'existence d'un prolongement $f' \in Y_0^X$ de f entraîne l'existence d'un prolongement $g' \in Y_0^X$ de g. En remplaçant la notion de diviseur par celle de diviseur en dimension m on obtient les définitions: des zéro fonctions, des fonctions conjuguées, des fonctions primitives et unitaires, ainsi que la définition d'ordre d'une fonction — notions analogues, mais limitées à la dimension m. On voit sans peine que la thèse du théorème du N° 3 et les thèses des corollaires qui le suivent restent valables lorsqu'on remplace la condition $f|g$ par la condition $f \underset{m}{|} g$ en supposant en outre que $m \geqslant \dim X_0 + 1$ et $m \geqslant \dim Y_0$.

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THEORETICAL PHYSICS

"Equivalent" potentials for the motion of the nucleon in pseudoscalar fields

by

J. WERLE

Presented by L. INFELD on December 30, 1954

An interesting consequence of the relativistic equations of motion of a particle in external potentials (fields) was discussed by the author in [1], [2]. It was shown there that all the relativistic equations considered include in a hidden form some additional forces of purely relativistic origin. These forces are significant only in the vicinity of the singularity of the potential. Whether they are of a repulsive or attractive character depends solely on the transformation properties of the potential. It has been shown that this statement is true for classical relativistic equations of motion as well as for Klein-Gordon and Dirac equations of the quantum theory.

This effect seems to be very important for the meson theory of nuclear forces, because in this case the additional forces are very strong and their range is almost of the same order of magnitude as the range of ordinary nuclear forces. It is interesting to note that even for attractive neutral scalar or pseudoscalar potentials the additional forces are strongly repulsive and therefore prevent the particle from coming too near to the singularity. Extension of these results in a quantitative manner on the interaction problem of many nucleons is, of course, not so straightforward but the effect exists probably and can considerably change our ideas about nuclear forces.

In both the above mentioned papers only neutral fields were taken into account. Now we shall discuss the most important case of a symmetrical mixture of charged and neutral pseudoscalar fields. Let us consider equations of motion for nucleons with pseudoscalar coupling

$$(1) \qquad i\hbar \frac{\partial \Psi}{\partial t} = \left\{ -i\hbar c \varrho_1 \vec{\sigma} \cdot \nabla + \varrho_3 mc^2 + g \varrho_2 \sum_{i=1}^{3} \tau_{(i)} \Phi_{(i)} \right\} \Psi,$$

Ψ is an eight-component wave function of the type Ψ_{ab}, where $a=1,2,3,4$ and $b=1,2$. $\Phi_{(i)}=(\Phi_{(1)}, \Phi_{(2)}, \Phi_{(3)})$ denotes combined charged and neutral pseudoscalar fields (symmetrical mixture). Symbols $\varrho_1, \varrho_2, \varrho_3, \sigma_k (k=1,2,3)$ denote the usual Dirac matrices acting of course only on the first index a of the function Ψ. $\tau_{(i)}=(\tau_{(1)}, \tau_{(2)}, \tau_{(3)})$ are isotopic spin matrices of the second rank acting on the second index b of Ψ. The difference in mass between proton and neutron is neglected.

Putting

$$(2)\qquad \Psi=\begin{pmatrix}\psi_{1b}\\ \psi_{2b}\\ \psi_{3b}\\ \psi_{4b}\end{pmatrix}=(\chi+\varrho_1\bar{\chi})e^{\frac{-imc^2t}{\hbar}};\qquad \chi=\begin{pmatrix}\psi_{1b}\\ \psi_{2b}\\ 0\\ 0\end{pmatrix}e^{\frac{imc^2t}{\hbar}};\qquad \bar{\chi}=\begin{pmatrix}\psi_{3b}\\ \psi_{4b}\\ 0\\ 0\end{pmatrix}e^{\frac{imc^2t}{\hbar}},$$

we can write equation (1) in the following form

$$(3)\qquad i\hbar\frac{\partial\chi}{\partial t}=\left(-i\hbar c\vec{\sigma}\cdot\nabla-ig\sum_{i=1}^{3}\tau_{(i)}\Phi_{(i)}\right)\bar{\chi},$$

$$(4)\qquad \left(i\hbar\frac{\partial}{\partial t}+2mc^2\right)\bar{\chi}=\left(-i\hbar c\vec{\sigma}\cdot\nabla+ig\sum_{i=1}^{3}\tau_{(i)}\Phi_{(i)}\right)\chi.$$

In (3) and (4) only the usual Pauli matrices of the second rank appear. Functions χ and $\bar{\chi}$ have four components: $\chi=(\chi_{ab})$, $\bar{\chi}=(\bar{\chi}_{ab})$, $a=1,2$; $b=1,2$.

If $\Phi_{(i)}$ do not depend explicitly on time, then the elimination of $\bar{\chi}$ or χ from (3) and (4) is very simple. For stationary solutions $i\hbar\frac{\partial}{\partial t}=\varepsilon=E-mc^2$ and without any restriction we obtain

$$(5)\qquad \left\{-\frac{\hbar^2}{2m}\Delta+\frac{\hbar g}{2mc}\vec{\sigma}\cdot\nabla\sum_{i=1}^{3}\tau_{(i)}\Phi_{(i)}-\frac{g^2}{2mc^2}\left(\sum_{i=1}^{3}\tau_{(i)}\Phi_{(i)}\right)^2\right\}\chi=\varepsilon'\chi,$$

where $\varepsilon'=\varepsilon\left(1+\frac{\varepsilon}{2mc^2}\right)$. For $\varepsilon\ll mc^2$ we can put $\varepsilon'=\varepsilon$ and now (5) reduces strictly to the non-relativistic Pauli equation

$$(6)\qquad \left\{-\frac{\hbar^2}{2m}\Delta+V\right\}\chi=\varepsilon\chi$$

with potential

$$(7)\qquad V=\frac{\hbar g}{2mc}\vec{\sigma}\cdot\nabla\sum_{i=1}^{3}\tau_{(i)}\Phi_{(i)}-\frac{g^2}{2mc^2}\left(\sum_{i=1}^{3}\tau_{(i)}\Phi_{(i)}\right)^2,$$

which we shall call "equivalent". Having a solution χ of (5) we obtain $\bar{\chi}$ from (4), i. e.

$$\bar{\chi}=\left|\frac{-i\hbar c\vec{\sigma}\cdot\nabla-ig\sum_{i=1}^{3}\tau_{(i)}\Phi_{(i)}}{2mc^2+\varepsilon}\right|\chi \tag{8}$$

without introducing any further integration constants. Therefore it can be said that the differential equation (5) in spite of its non-relativistic form is strictly equivalent to (1) (of course together with definitions (2) and (8)).

From the form of the "equivalent" potential (7) we see that the additional relativistic force described by the second quadratic term is always repulsive. When the functions $\Phi_{(i)}$ contain a singularity of the type $1/r^n$ $(n>1)$, then at short distances from the singularity the repulsive relativistic force will always prevail over the ordinary one. The potentials $\Phi_{(i)}$ can of course be created by another particle, which we denote by number 2, distinguishing it in this way from particle 1, the motion of which we are just considering. Neglecting in the first approximation the retardation effect of the meson field, we shall take well-known "instantaneous" values of these potentials

$$\Phi_{(i)}=\frac{\hbar g}{2mc}\tau_{(i2)}\vec{\sigma}_2\cdot\nabla\frac{e^{-\mu r}}{r}. \tag{9}$$

Inserting the quasistatic expressions (9) into (7) we obtain the following „equivalent" interaction potential.

$$V=\frac{a^2}{3}\sum_{i=1}^{3}\tau_{(i1)}\tau_{(i2)}\{\vec{\sigma}_1\cdot\vec{\sigma}_2+S_{12}\cdot R(r)\}\frac{e^{-\mu r}}{r}+\frac{g^2a^2}{2mc^2}\left(\sum_{i=1}^{3}\tau_{(i1)}\tau_{(i2)}\right)^2Q(r)\frac{e^{-2\mu r}}{r^2} \tag{10}$$

where we have put

$$\begin{aligned}S_{12}&=\frac{3(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^2}-\vec{\sigma}_1\cdot\vec{\sigma}_2; \qquad R(r)=1+\frac{3}{\mu r}+\frac{3}{\mu^2r^2}.\\ Q(r)&=\left(1+\frac{1}{\mu r}\right)^2, \qquad a=\frac{\mu\hbar g}{2mc}.\end{aligned} \tag{11}$$

The first part of expression (10) proportional to g^2 is well known. It represents spin-dependent non-central interaction, which was very often encountered in various meson theories of nuclear forces. Nevertheless it is inadequate, because it contains — if taken alone — a strong attractive $1/r^3$ singularity excluding thus the existence of reasonably determined stationary states of the deuteron. But from the deduction of (5) we see that there is no reason to neglect the second part of the potential (10) which is repulsive in all states and contains a still stronger $1/r^4$ singularity. Thus, at sufficiently short distances the repulsive term will

always prevail over the attractive $1/r^3$ singularity. Taking account of this repulsive term, we have no difficulties connected with high singularities in the interaction potential. From (10) it can be seen that the repulsive force, though central and spin independent, depends on isotopic spins of both particles. For states antisymmetric in isotopic spins $\sum_{i=1}^{3} \tau_{(1)} \tau_{(1)} = -3$ and we have then the greatest repulsive force.

Of course we cannot claim the validity of potential (10) for the two-body problem. In order to solve the two-body problem the nucleons must be treated from the beginning in a more symmetrical manner. Also the higher order effects in the coupling constant g must be taken properly into account. But at any rate the results obtained here and in [1], [2] cast some new light on the problem of the origin of repulsive forces between nucleons. The form in which the additional force appears in [7] suggests that the effect of repulsion can be independent of the sign of higher order in g^2 contributions to the interaction potential, no matter how singular they are.

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THEORETICAL PHYSICS

Note on the $^6\mathrm{Li}(n,d)^5\mathrm{He}$ reaction

by

J. DĄBROWSKI and J. SAWICKI

Presented by W. RUBINOWICZ on December 20, 1954

The $^6\mathrm{Li}(n,t)^4\mathrm{He}$ reaction was recently investigated (Dąbrowski, Sawicki 1954) from the point of view of a simple two-body "$\alpha - d$" model of the ^{6}Li nucleus.

Frye [2] gives experimental angular distributions for both reactions. $^6\mathrm{Li}(n,d)^4\mathrm{He}$ and $^6\mathrm{Li}(n,t)^4\mathrm{He}$, for 14 MeV neutrons and compares the absolute values of the respective differential and total cross-sections. It appears that the absolute value of $(d\sigma/d\Omega)_{\Theta=0}$ for $^6\mathrm{Li}(n,d)^5\mathrm{He}$ is about 5.1 times greater than that for $^6\mathrm{Li}(n,t)^4\mathrm{He}$.

In the present note we attempt to give a theoretical explanation of these facts with the help of our "$\alpha - d$" model. On applying the usual Born approximation as in [1] with neglect of the incident neutron-α particle interaction we obtain the differential cross-section for the $^6\mathrm{Li}(n,d)^5\mathrm{He}$ reaction in the c. m. system:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi\hbar^2)^2} M_0^* M_5^* \frac{k_d}{k_0} \frac{1}{2\cdot 3} \sum_{\substack{\mu_0\mu_a \\ \mu_5\mu_d'}} \left| \int d\xi\, d\vec{r}_{ad}\, d\vec{r}_{d0}\, d\vec{\varrho}\, d\sigma_0 d\sigma_n d\sigma_p\, \chi^{*}_{\mu_d'} \chi^{*}_{\mu_5} e^{-i\vec{k}_\alpha\left(\vec{r}_{ad} - \frac{1}{5}\vec{r}_{a0}\right)} V_{0d} \psi_0 \right|^2 . \tag{1}$$

where $M_5^* = 10/7\,M$, $M_0^* = 6/7\,M$ are resp. the ^{5}He and the incident neutron reduced mass (M = nucleon mass); $\hbar\vec{k}_0$, $\hbar\vec{k}_d$ are the neutron and deuteron momenta in the c. m. system: μ_0, $\mu_d'(\mu_{d'})$, μ_5 are the resp. neutron, deuteron and ^{5}He magnetic numbers: $\chi_{\mu_d} = S_{\mu_d}(\sigma_n\sigma_p)\psi(\varrho)$ is the deuteron internal wave function, $\chi_{\mu_5} = \Phi_{\mu_5}(\vartheta_{a0}\varphi_{a0}\sigma_0)R(r_{a0})\chi_a(\xi)$ is the $P_{3/2}$ state ^{5}He nucleus internal wave function;

$$\psi_0 = \chi_{\mu_0}(\sigma_0) e^{i\vec{k}_0\vec{r}_0} \chi_{\mu_d}\chi_a(\xi)\Phi(r_{ad}), \tag{2}$$

where $\chi_a(\xi)$ is the internal wave function of the α-particle, $\Phi(r_{ad})$ is the S-state "$\alpha+d$" system internal function; σ are the respective spin va-

riables, $\vec{r}_0$ - the vector from ^{6}Li to the incident neutron, $\vec{r}_{a0}$ - the vector from α-particle to the neutron, and $\vec{r}_{ad}$ – the vector from α particle to the deuteron; V_{0d} is the interaction between the incident neutron and the deuteron.

We assume the simplest form of V_{0d}:

$$V_{0d} = -\frac{3\pi a_{nd}\hbar^2}{M}\delta(\vec{r}_{ad}-\vec{r}_{a0}) \tag{3}$$

- the Fermi "contact pseudopotential" with $a_{nd}=\sqrt{\frac{2}{3}a_4^2+\frac{1}{3}a_2^2}=5.08\cdot 10^{-13}$ cm. – the $n-d$ "scattering length" given by Christian and Gammel (1953)*). This interaction is consistent with the assumed model deuteron being treated as an individual particle. This corresponds to the simple "knock-out" mechanism of the whole deuteron by the incident neutron the latter being captured by the α particle in a way similar to the mechanism of the (n,p) reaction proposed by Austern, Butler and McManus (1953). Evidently this is a crude assumption since the deuteron is a fairly loose structure. In more exact calculations V_{0d} would be used in the form of the sum of the interactions between the incident neutron and each of the nucleons of the deuteron separately. In this way besides the "knock-out" mechanism we would have the pickup of the proton by the incident neutron. However, the contribution of the latter mechanism to the angular distribution is decreased by the also omitted contribution coming from the incident neutron-α particle interaction $V_{0\alpha}$. On taking into account $V_{0\alpha}$ we have the pickup mechanism of the α particle from ^{6}Li by the incident neutron causing an increase in the number of backward scattered deuterons, while the ordinary pickup gives an increase in the number of forward scattered deuterons.

Under our assumptions we have, after integrations and summations:

$$\frac{d\sigma}{d\Omega}=\frac{2}{\pi\hbar^4}M_0^*M_5^*\frac{k_d}{k_0}\left(\frac{3\pi a_{nd}\hbar^2}{M}\right)^2\left|\int_0^\infty j_1(qr)R(r)\Phi(r)r^2dr\right|^2. \tag{4}$$

where j_1 is the spherical Bessel function of order 1;

$$q^2=\frac{16}{25}k_d^2+\frac{4}{9}k_0^2-\frac{16}{15}k_dk_0\cos\theta, \tag{5}$$

θ being the scattering angle in the c. m. system.

We assume the ^{5}He nucleus radial function $R(r)$ to be:

$$R(r)=\begin{cases}\dfrac{\pi}{a\sqrt{3}}\dfrac{1}{\sqrt{r}}J_{3/2}\left(\dfrac{\pi r}{a}\right) & \text{for } r<a\\[2ex] \sqrt{\dfrac{2a}{3}}\dfrac{1}{r^2} & \text{for } r>a.\end{cases} \tag{6}$$

*) The assumption of zero range interaction for a neutron energy of about 10 MeV must be considered as a rough approximation only.

appropriate to a particle in a P-level of zero total energy in a square well of width a, used by Hughes and Le Couteur [5]. The value $a = 2.9 \cdot 10^{-13}$ cm. suggested by Adair (1952), is used. Similarly to [1], $\Phi(r)$ is the "$a+d$" system function for the $a-d$ interaction in the form of a square well of radius R_0.

The resulting angular distribution for 14 MeV neutrons was obtained from [4] with the help of graphical integrations.

TABLE

Theoretical and experimental values of $(d\sigma/d\Omega)_{\theta=0}$ for (n, d) and (n, t) reactions for 14 MeV neutrons (for (n, t) the best values of [1] are given).

	Theoretical values of $(d\sigma/d\Omega)_{\theta=0}$ in $\frac{\mu b}{\text{ster.}}$			Exp. values [2]
	$R_0 = 4 \cdot 10^{-13}$ cm,	$R_0 = 6 \cdot 10^{-13}$ cm.	$R_0 = 7.5 \cdot 10^{-13}$ cm.	
^{6}Li$(n,d)^5$He	71.9	66.2	78.8	38
^{6}Li$(n,t)^4$He	6	1.6		7.5

The absolute values of $(d\sigma\, d\Omega)_{\Theta=0}$ for different values of R_0 for both reactions: ^{6}Li$(n,d)^5$He and ^{6}Li$(n,t)^4$He compared with the respective experimental values are indicated in the Table.

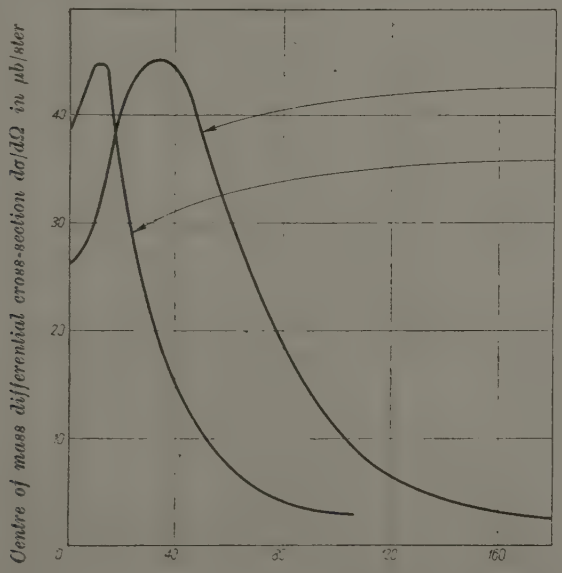

Fig. 1. Differential cross-section for ^{6}Li (n, d) ^{5}He reaction

While for the ^{6}Li$(n,t)^4$He reaction the best angular distribution was obtained for $R_0 \sim 6.10^{-13}$ cm., in this case we get better agreement

with the experimental results for larger values of R_0. The angular distribution for the $^6\mathrm{Li}(n,d)^5\mathrm{He}$ reaction for $R_0 = 7.5 \cdot 10^{-3}$ cm. is given in Fig. 1. For smaller values of R_0, the maximum of the curve becomes broader and displaced somewhat towards larger angles.

We observe that, in spite of our simplifying assumptions, general agreement in regard to the character of the theoretical and experimental angular distributions was obtained for the $^6\mathrm{Li}(n,d)^5\mathrm{He}$ reaction without taking into account the "ordinary" pickup mechanism.

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Unitary transformations and spinor calculus

by

L. INFELD and J. PLEBAŃSKI

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1. In a previous paper [1] we considered linear unimodular transformations of non-commutative *real* operators. The transformations leave the commutation relations invariant, hence they are a special case of unitary transformations. In that paper we were specially interested in finding a unitary operator which would induce a given unimodular transformation. The mathematical formalism of *real* spinors in a *three*-dimensional Minkowski-space is strictly connected with this problem. The mathematical formalism developed in the previous paper may be useful in quantum mechanics; it has been used, in particular, by one of us for the construction and discussion of generalized Kennard-packets [2].

However, the similar problem of linear unimodular transformations for *complex* operators is much more important. Indeed most of the important expressions in quantum field theory, especially in quantum electrodynamics, are formed from such operators. We shall show that by considering the linear transformations leaving the commutation relations invariant, one immediately arrives at an application of spinor calculus [3]. In addition, by considering the unitary operators as inducing linear unimodular transformations for spinors, one arrives in a natural way at the formalism connected with the four-dimensional Minkowski-space.

The purpose of the present paper is to give an outline of the main mathematical ideas of the theory mentioned above, which can be understood without knowing the contents of the previous paper [1]. The greater part of paper [1] follows as a special case of a more general theory dealing with complex operators.

2. It is known that for the Fourier-coefficients of a complex field $\Psi(x), q_k$ and for the corresponding Fourier-coefficients of a field $\pi(x), p_k$ canonically conjugate with Ψ, where q_k, p_k are the complex operators,

we have the commutation relations

$$[p_k, q_{k'}] = \frac{\hbar}{i}\delta_{kk'} = [p_k^*, q_{k'}^*], \tag{1}$$

and the commutators from all other pairs of variables vanish [4].

Considering only one pair of operators p_k, q_k with fixed k it can easily be seen that, introducing $\xi_1 \overset{\text{df}}{=} q_k/a$, $\xi_2 = p_k/b$, where $ab = \hbar$, and adopting from spinor calculus the well known notation $(\xi_a)^* = \xi_{\dot a}$, the commutation relations (1) may be rewritten in the form

$$[\xi_a, \xi_\beta] = i\gamma_{a\beta}, \qquad [\xi_a, \xi_{\dot\beta}] = 0 \tag{2}$$

($[\xi_{\dot a}, \xi_{\dot\beta}] = i\gamma_{\dot a\dot\beta}$ follows from (2)), where $\gamma_{a\beta}$ denotes the matrix: $\gamma_{a\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

From the very possibility of writing (1) in form (2) it is immediately seen that the operators ξ_a can be considered as non-commutative spinors; from the form of (2) it is clear that it remains invariant with respect to the linear unimodular transformations in the space ξ_a. Thus in the theory of unitary transformations of the operators ξ_a we can take advantage of the known spinor calculus. Let us recall some results from the spinor calculus, using the notation of Corson's work [3].

The matrices

$$g_{k\dot a\beta} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]^*), \tag{3}$$

and the matrices of spinor tensor

$$S^{kl\alpha\beta} = \tfrac{1}{2}(g^{k\dot\sigma\alpha} g^{l}{}_{\dot\sigma}{}^{\beta} - g^{l\dot\sigma\alpha} g^{k}{}_{\dot\sigma\beta}), \tag{4}$$

are of fundamental importance in spinor calculus.

It is known that $g_{k\dot\alpha\beta}$ may be used to connect a real four-vector C_k with a Hermitean spinor $C_{\dot\alpha\dot\beta}$; now $S^{kl\alpha\beta}$ may be used to connect with a symmetrical spinor $C_{\alpha\beta}$ a selfdual antisymmetrical (complex) tensor C_{kl}. If $C_{\dot\alpha\dot\beta}$, $C_{\alpha\beta}$ transform according to a unimodular transformation in the spinor-space, then C_k and C_{kl} transform according to a corresponding Lorentz-transformation induced by the unimodular transformation.

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*) The Greek indices take the values 1 and 2, the Latin indices run from 0 to 3. For both, summation over the dummy indices is understood. The Latin indices may be raised by the means of

$$g^{kl} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and the Greek indices by the means of $\gamma^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \gamma_{\alpha\beta}$, according to the rule $\eta^\alpha = \gamma^{\alpha\beta}\eta_\beta$.

3. Assuming for the spinors ξ_α the commutation relations (2), we shall now introduce some new operator quantities and we shall give their principal properties.

Let

(5) $$x_k = \tfrac{1}{2} g^{k\dot\alpha\beta} \xi_{\dot\alpha} \xi_\beta,$$

(6) $$F^{kl} = S^{kl\alpha\beta} \xi_\alpha \xi_\beta.$$

It can easily be proved that the tensor-operator F^{kl} is selfdual, i. e.,

$$\check F^{kl} = \tfrac{1}{2} g^{-1/2} \varepsilon^{klnm} F_{nm} = F^{kl}, \qquad g = \mathrm{Det} \| g_{kl} \| = -1.$$

Let us form from F^{kl} the tensor-operator $f_{kl} = \frac{1}{2}(F_{kl} + F^*_{kl})$, all components of which are real operators. The tensor $\check f_{kl} = \frac{1}{2}(F_{kl} - F^*_{kl})$ will be dual to f_{kl}.

It can easily be shown that the operators defined above have the following properties:

(7) $$x_k x^k = -\tfrac{1}{2},$$

(8) $$[x^k, x^l] = (i/2) f^{kl},$$

(9) $$[\xi^\alpha, x^k] = (i/2) g^{k\dot\nu\alpha} \xi_{\dot\nu}, \qquad [\xi^{\dot\alpha}, x^k] = (i/2) g^{k\dot\alpha\nu} \xi_\nu$$

(10) $$\tfrac{1}{4} F_{kl} F^{kl} = -3$$

(11) $$\tfrac{1}{4} F^*_{kl} F^{kl} = 0$$

(12) $$\tfrac{1}{4} f_{kl} f^{kl} = -\tfrac{3}{2}$$

(13) $$\tfrac{1}{4} f_{kl} \check f^{kl} = 0$$

(14) $$[F_{kl}, F_{nm}] = 4i(g_{km} F_{ln} - g_{lm} F_{kn} + g_{ln} F_{km} - g_{kn} F_{lm}),$$

(15) $$[f_{kl}, f_{mn}] = 2i(g_{km} f_{ln} - g_{lm} f_{kn} + g_{ln} f_{km} - g_{kn} f_{lm}),$$

(16) $$[f_{kl}, \check f_{mn}] = 2i(g_{km} \check f_{ln} - g_{lm} \check f_{kn} + g_{ln} \check f_{km} - g_{kn} \check f_{lm}),$$

(17) $$[f_{kl}, \xi^\alpha] = -i S^{\alpha\beta}_{kl} \xi_\beta, \; [f_{kl}, \xi^{\dot\alpha}] = -i S^{\dot\alpha\dot\beta}_{kl} \xi_{\dot\beta}.$$

Using the operators x^k, F^{kl}, f^{kl} and $\check f^{kl}$, we can easily construct a unitary operator leading to a general linear transformation of the operators ξ_α, leaving invariant relations (2). However in this note we shall limit ourselves to a consideration only of linear unimodular transformations which do not mix the dotted and undotted quantities.

4. Let A_{kl} be a real numerical antisymmetrical tensor, and $a_{\alpha\beta}$ a symmetrical spinor connected with A_{kl}, i. e.,

(18) $$a_{\alpha\beta} = \tfrac{1}{8} S_{kl\alpha\beta} A^{kl} \qquad A_{kl} = S^{\alpha\beta}_{kl} a_{\alpha\beta} + S^{\dot\alpha\dot\beta}_{kl} a_{\dot\alpha\dot\beta}.$$

If we denote $a_{\alpha\beta}=\begin{pmatrix}\alpha & \gamma\\ \gamma & \beta\end{pmatrix}$, then it may easily be shown that

$$a_{\alpha\nu}a^{\nu\beta}=\omega^2\delta_\alpha^\beta, \tag{19a}$$

$$\omega^2=\tfrac{1}{2}a_{\alpha\beta}a^{\alpha\beta}=\alpha\beta-\gamma^2 \tag{19b}$$

$$A=\tfrac{1}{4}A_{kl}A^{kl}=4(\omega^2+\omega^{2*}), \tag{20a}$$

$$iB=\tfrac{1}{4}A_{kl}\overset{\vee}{A}{}^{kl}=4(\omega^2-\omega^{2*}), \tag{20b}$$

so that ω, defined by (19b), may be written in terms of the invariants of the tensor A_{kl} as $\omega=[(A+iB)/8]^{1/2}$.

Let us consider now the real operator

$$T\overset{\text{df}}{=}\tfrac{1}{8}A^{kl}f_{kl}\equiv\tfrac{1}{2}(a^{\alpha\beta}\xi_\alpha\xi_\beta+a^{\dot\alpha\dot\beta}\xi_{\dot\alpha}\xi_{\dot\beta}). \tag{21}$$

It has the following properties:

$$[T,\xi_\alpha]=-ia_\alpha{}^\beta\xi_\beta,\qquad [T,\xi_{\dot\alpha}]=-ia_{\dot\alpha}{}^{\dot\beta}\xi_{\dot\alpha}. \tag{22}$$

Let us consider a unitary transformation of operators ξ_α by the aid of the unitary operator e^{iT}: $\xi'_\alpha=e^{iT}\xi_\alpha e^{-iT}$ (it follows that $\xi'_{\dot\alpha}=e^{iT}\xi_{\dot\alpha}e^{-iT}$). To calculate ξ'_α we shall introduce an auxiliary quantity $\eta_\alpha=e^{i\lambda T}\xi_\alpha e^{-i\lambda T}$, where λ is a real number. According to (22) we shall find at once that η_α have to fulfil the differential equations $\dfrac{d\eta_\alpha}{d\lambda}=a_\alpha^\beta\eta_\beta$, formally integrating the equations by the initial condition $\eta_\alpha|_{\lambda=0}=\xi_\alpha$ and then putting in the resulting expression $\lambda=1$ we easily find ξ'_α as

$$\xi'_\alpha=t_\alpha{}^\beta\xi_\beta, \tag{23a}$$

where

$$t_\alpha{}^\beta=\cos\omega\,\delta_\alpha^\beta+\frac{\sin\omega}{\omega}a_\alpha^\beta. \tag{23b}$$

We find the transformation formulas for ξ_α by the conjugation in (23a). It may easily be seen that the matrix $t_\alpha{}^\beta$ from (23b), containing 6 parameters, is unimodular, i. e., $\mathrm{Det}\|t_\alpha^\beta\|=1$.

So, with respect to the unitary transformation connected with e^{iT} the quantities with Latin indices transform according to a Lorentz-transformation induced by the unimodular transformation (23).

We see from (23) that a definite unimodular transformation with matrix $t_\alpha{}^\beta$ corresponds to a unitary operator e^{iT}, with T given by (21). Conversely, for a given most general unimodular transformation with

matrix $t_{\alpha\beta}$ such a T can be found and also such an $a_{\alpha\beta}$ that the operator e^{iT} leads to the transformation. This and similar problems will be the subject of another paper*).

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*) A more detailed paper on the same subject will be published in Acta Phys. Pol

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EXPERIMENTAL PHYSICS

Use of electric field pulses as a method of investigating semiconducting films

by

H. RZEWUSKI and L. SOSNOWSKI

Presented by A. SOŁTAN on December 30, 1954

The properties of semiconductors are essentially dependent on the concentration and mobility of current carriers, on their life-time and on the density of surface states. The application of these properties is of great importance in transistor physics.

There are two methods widely in use for determining the previously mentioned quantities, namely, that based on the Hall effect of measuring the mobility and concentration of majority current carriers, and the Haynes-Shockley method for determining drift velocity.

All the methods used for investigating the properties of surface states are dependent on contact phenomena, on the modulation of conductivity by an electric field and on the photo-voltaic effect.

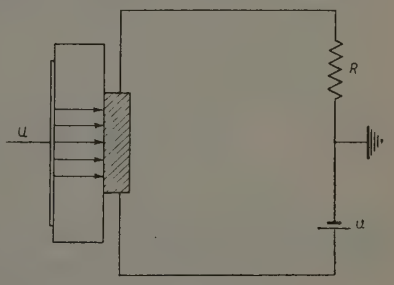

Fig. 1

As was found by Shockley, the conductivity of germanium may be modulated by applying an electric field to the layer [1], [2]. In this work a new method has been developed, based in principle on this effect. It allows simultaneous measurement both of the concentration and mobility of majority current carriers and of the density of surface states.

The existence of surface states on the free surface of a crystal, predicted theoretically by Tamm, has been established in several experiments reported recently. We are now in a position to understand better the surface levels arising from the crystal boundaries. The electrical charge induced to the surface of a semiconductor may be partly localized by the surface levels since the latter are not filled up with current carriers (holes or electrons).

Let us consider a semiconducting film of the n-type, evaporated onto an insulating plate, e. g. one made of glass (Fig. 1). The surface

states will appear on the surface of the layer in contact with the plate. If a metal electrode parallel to the layer is brought into contact with the opposite side of the glass and a field is applied to it, then the electrical charge n_v will be induced to the layer.

The surface of the layer is not restricted to the geometrically monoatomic Tamm surface layer but must be understood in a wider sense as including a deeper region penetrating into the semiconductor. Some of the charge density n_s is fixed in the surface states, and what remains, $n_p = n_v - n_s$, goes into the conduction band of the semiconductor changing the electron concentration which is present there.

The layer considered works as a detector, i. e. the ohmic contacts of a large area produced, for example, by evaporation are brought into

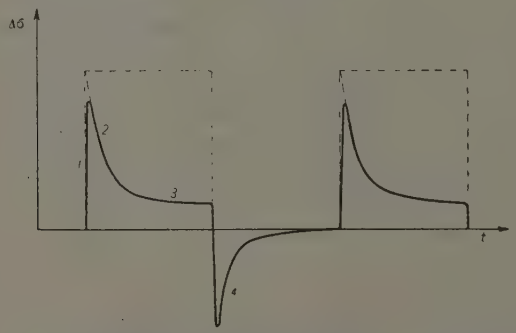

Fig. 2. Diagram showing the changes in the conductivity of the layer

contact with the layer, and the external circuit consists of a resistance R and a supply battery U. When a voltage is applied to the circuit, a current I begins to flow, dependent not only on the battery voltage U and the load resistance R, but also on the conductance of the layer $\sigma = e.n.u$. It may readily be observed that the additional charge in the conduction band through changes in electron concentration $n_0 + n_p$ results in a change of conductance $\sigma' = e(n_0 + n_p)u$. Changes in the circuit current are registered by changes in the voltage drop across the resistance R.

In the present method the rectangular pulses of an electric field are applied to the layer.

Simultaneously with the variations in the field there occurs a variation in the conductance of the layer. This may be observed on the oscilloscope screen. The changes in the conductivity of the layer differ from the rectangular form and are shown in Fig. 2.

On applying the electric field to the layer in the form of rectangular pulses, some current carriers are induced on the semiconductor. If there were no surface states on the surface of the layer, all the induced

current carriers would go into the conduction band. This would change the conductance of the layer (dotted line) in accordance with the changes in the field. The patterns observed on the screen (unbroken line) result from the interaction of the surface states in which some charge is bound. The curve is observed to increase [1]; it does not, however, take on a maximum value (the peak of the dotted impulse). The current carriers induced on the material are immediately immobilized by the surface states. If the steep rise of the applied voltage corresponded to zero time of increase, the curve representing the change in conductance would coincide with the dashed line. The additional charge in the surface states changes the distribution of the current carriers in the barrier layer just beneath the surface. The height and thickness of the barrier are also changed. As seen from Fig. 2, the process of establishing equilibrium in the barrier layer between the surface states charge and the space charge leads to a fixed value of the conductance. The value (3) corresponds to the number of carriers left in the conduction band under fixed conditions.

After switching off the field, an electric charge containing a number of current carriers in the surface states pushes back an equal number of electrons from the conduction band. This results in a considerable change of the potential barrier, and subsequently in an emptying of the surface states. Equilibrium between the surface state and the potential barrier is similarly attained. The character of these changes gives evidence of the same mechanism at work in establishing the equilibrium state.

By extrapolating segment (2) of the curve in Fig. 2 we eliminate the influence of the surface states.

The point of intersection corresponds to the impulse height, which is indicated by a dotted line. Once we know the value n_v of the full charge induced by the electric field, we can calculate the concentration n_0 of the current carriers in the film. Then the ratio of the value corresponding to a fixed state (3) to the previously mentioned value allows the degree of modulation conductivity m of the layer to be determined:

$$m = \frac{\Delta\sigma}{\sigma} = \frac{e(n_v - n_s)u}{en_0u} = \frac{n_v - n_s}{n_0} = \frac{n_p}{n_0}.$$

This value gives a basis for determining the density of the surface states. Knowing the conductivity of the layer and the concentration of the current carriers as determined above, we may easily find their mobility. By examining the direction of the conductivity changes, it may be determined whether the layer is of the n- or p-type. If, when electrons are introduced into the layer, the conductivity is increased, this proves that we have to do with an n-type of semiconductor. Otherwise the layer is of the p-type.

More detailed analysis of the curve and particularly of its segment (2) furnishes further information concerning the time taken to establish equilibrium between the potential barrier and the filling of the surface states.

The experimental arrangement in the present method may be described as follows (Fig. 3).

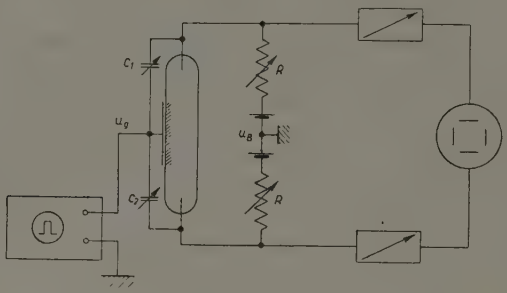

Fig. 3

The cell provided with the semiconducting film under investigation works in a circuit which consists of matched resistance $R_0 \cong 2R$ and a supply battery at a voltage U_B. The surface electrode is supplied with rectangular pulses at a voltage U_g. The voltage variations on the load resistance $(R+R)$, corresponding to the modulation of the conductance of the layer are then amplified by two indetical wide-band amplifiers and are applied to the vertical plates of an oscilloscope. The deviation on the screen corresponds to the sum of values of both amplified voltages.

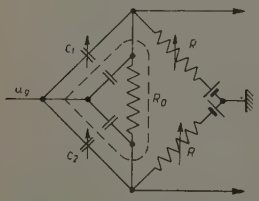

Fig. 4

The bridge system used allows compensation of the voltages which disturb the measurements. These voltages are due to the capacity coupling between the load circuit of the cell and the output of a square-wave generator. The harmful capacities are those between the applied electrode A and the contact electrodes to the cell (Fig. 4).

In view of the different values of the capacities mentioned, the amplitudes of both harmful pulses may be controlled by varying both the resistances R. The phase tuning of both voltages is performed by means of additional regulated capacities C_1 and C_2 and then, as these voltages have an opposite phase, no pulse will be observed on the oscilloscope screen.

The problem of compensation is essential here as we may judge from the rough data concerning several voltages obtained in certain measurements.

The effect measured was of the order of several millivolts and the harmful voltages were greater by three orders of magnitude, i. e. as high as several volts.

So far the measurements have been made on microcrystalline films of PbS evaporated *in vacuo* in closed envelopes made of glass between two ohmic contacts made of graphite [3].

The experimental conditions and the measurements for one of the PbS films are given below.

Characteristics of the square-wave generator:

Voltage	$U = 100$ V
Frequence of repetition	$f = 200$ c/s
Time of impulses (in parts of a period)	$k = 1/2$

Characteristics of the PbS layers:

Thickness of layer	$a = 10^{-4}$ cm.
Length of layer (distance between contacts)	$b = 0.2$ cm.
Width of layer	$c = 1$ cm.
Thickness of envelope walls	$d = 0.05$ cm.
Resistance of layer	$R = 180$ $k\Omega$

The results obtained:

Type of layer	p
Concentration of current carriers	$p = 10^{17}$ cm^{-3}
Mobility of current carriers	$u = 0.7 \frac{\text{cm}^2}{\text{V sec}}$
Activity of surface states	$s = \frac{n_s}{n_v} = 75\%$

In this work an outline of a new method has been presented together with the first results. A fuller interpretation, as well as further development of the apparatus and its application to other substances, is now being worked out.

The present work was presented at a scientific meeting of Polish physicists held at Spała on September 1—15, 1954.

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CHEMISTRY

The influence of catalytic poisons and of substances increasing hydrogen overvoltage on the electrolytic reduction yield of nitrobenzene and oxalic acid

by

M. ŚMIAŁOWSKI and H. JARMOŁOWICZ

Presented by M. ŚMIAŁOWSKI on December 20, 1954

The views held until now concerning the mechanism of the processes of cathodic reduction of organic compounds are not fully established. The majority of authors are of the opinion that in the case of cathodes with low hydrogen overvoltage, such as Pt, Pd, Fe and Ni, the actual agent of reduction is atomic hydrogen; whereas in the case of cathodes with high overvoltage, as e. g. Hg and Pb, the reduction probably arises in consequence of the action of electrons supplied by the cathode and of protons collected from the solution [1]. This problem is still, however, the subject of continual research and discussion [2] [3] [4], and the criteria for determining whether particular reactions belong to the one or the other type of cathodic reduction processes have not yet been fully defined.

It is reasonable to suppose that negative catalyzers, such as P, S, As, Sb, Se, Te, which prolong the period of existence of the atomic form of hydrogen in contact with the cathode [5], should facilitate reactions of the first type. Such catalyzers, on the other hand, should remain without positive influence on processes of the second kind which are in large measure dependent on the value of the hydrogen overvoltage. In order to test the correctness of this assumption, comparative examinations were made of two known examples of reduction of organic compounds: nitrobenzene and oxalic acid.

Reduction of nitrobenzene to aniline

Electrolytic reduction of nitrobenzene is dependent on the conditions under which electrolysis is carried out. This process passes through a series of transition stages, and gives various end products [6]. The greatest yields of aniline are obtained on cathodes made of copper, zinc, tin, and

lead, while phenylhydroxylamine is mainly formed on cathodes made of platinum, nickel, and iron. These facts and also the circumstance stated by Glasstone [7], viz., that the yield of aniline increases in proportion as the temperature is raised, seem to indicate that in the process of nitrobenzene reduction the value of the hydrogen overvoltage is, at most, a factor of secondary importance. The favourable action of a copper cathode on the yield of aniline is attributed to the catalytic capabilities of this metal. The data concerning the influence of catalytic poisons on the reduction of nitrobenzene and other organic compounds are scanty and for the most part contradictory. Stackelberg and Weber [4] noted a high reduction yield of nitrobenzene on a Pd cathode poisoned with arsenic but came to the conclusion that free hydrogen atoms do not take part directly in this process, only electrons and protons.

In the course of our experiments we distinctly noted the positive influence of catalytic poisons, especially phosphorus, arsenic and sulphur, on the yield of aniline, both in the process of nitrobenzene reduction by zinc powder and in the case of electrolysis on iron or lead cathodes. On the other hand, substances increasing the hydrogen overvoltage did not influence the yield of the reduction process.

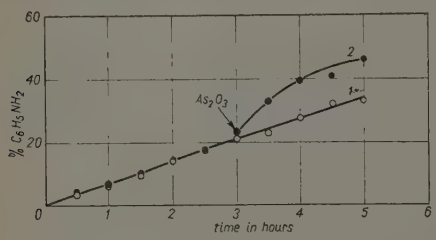

Fig. 1

Figure 1 represents, by way of example, the course of two experiments carried out, one without (curve 1), and one with the addition of arsenic trioxide (curve 2). Both curves show the dependence of aniline yield on duration of electrolysis. In the second experiment (curve 2) arsenic trioxide (in quantity 0.02 g./l. of cathodic solution) was added exactly 3 hours after the commencement of electrolysis, at the moment indicated on the diagram by an arrow. A considerable increase in the aniline yield is apparent after the addition of As_2O_3. The condition of these experiments were as follows: cathode – a sheet of lead, 1 dm^2. in area; cathodic liquid – 10 g. of nitrobenzene dissolved in 15 ml. of ethyl alcohol and mixed with 400 ml. of an aqueous solution of 6% HCl; cathodic current density – 5 amps./dm^2.; anode – a sheet of lead in a solution of 10% H_2SO_4; temperature – from 20 to 50°C; a finely-grained ceramic diaphragm separated the cathodic compartment from the anode. The aniline content was determined titrimetrically by diazotisation, with a 0.1 N solution of sodium nitrite, of samples taken every half-hour from the cathode compartment.

In Figure 2 the results are presented of another series of experiments, in which the cathodic liquid was a homogeneous solution of 10 g.

of nitrobenzene in 280 ml. of 80% acetic acid and 140 ml. of 25% hydrochloric acid.; except for this the conditions were the same as before. Curve 1 was obtained in the absence of poisons, curve 2 — in the presence of 0.02 g./l. As_2O_3, and curve 3 — in the presence of 1.2 g./l. of yellow phosphorus, dissolved in carbon disulphide. The sulphur compounds: Na_2S, H_2S or thiourea, produced a somewhat smaller increase in the yield of aniline than did the arsenic compounds, and the influence of antimony, selenium and tellurium was weaker still. The classification of catalytic poisons, from the point of view of their ability to increase the yield of aniline, seems therefore in general outline to conform with that specified in the case of the diffusion of hydrogen into steel [5]. On account, however, of the divergence in the results of the experiments, an exact estimation of the quantitative differences in the respective actions of particular poisons was not possible.

The results obtained on iron cathodes were similar to those obtained on lead ones, but the divergences in this case were even more serious.

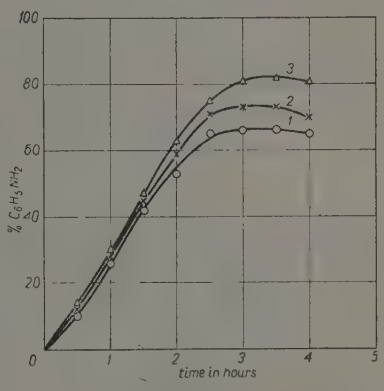

Fig. 2

The reduction of oxalic acid to glyoxalic acid

In contrast to nitrobenzene, the reduction of oxalic acid seems to be closely connected with the value of the hydrogen overvoltage. As early as 1904, Tafel and Friedrichs [8] observed that the yield in the reduction of oxalic acid to glyoxalic acid was greater on a mercury cathode than on other metals. This yield diminishes in proportion as the temperature rises [9] and is clearly dependent on the course of the polarisation curves [10]. Thus it can be assumed that catalytic poisons do not cause an increase in the yield of glyoxalic acid.

In the course of our experiments on the electrolytic reduction of oxalic acid we applied a method described by Izgarishev and Ariamova [10]: cathode — a sheet of lead 0.85 dm^2. in area; cathodic solution — 10 g. of crystalline $(COOH)_2.2H_2O$ in a 4% H_2SO_4 solution; cathodic current density — 2.35 amps./dm^2.; anode — lead in an 11% H_2SO_4 solution; the cathode compartment was separated from the anodic by a finely grained ceramic diaphragm; the temperature ranged from 10 to 12°C; the glyoxalic acid content was determined by the sodium disulphide method.

Figure 3 represents by way of example the course of three experiments, the first of which (curve 1) was carried out in the above-men-

tioned solution, without any additions, the second (curve 2) in the presence of the addition of 0.02 g. of As_2O_3 per litre, and the third (curve 3) in a solution containing 25 g. of sodium sulphate per litre. As can be seen from this diagram, the addition of arsenic trioxide, which lowers the hydrogen overvoltage, results in a considerable fall in the yield of glyoxalic acid; in contrast to this, the addition of sodium sulphate, which raises the hydrogen overvoltage on a Pb cathode, causes an increase in the yield.

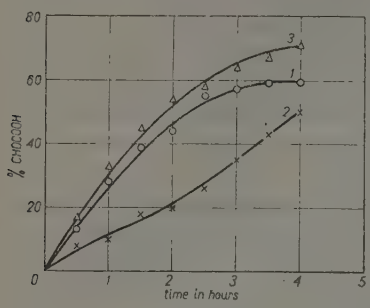

Fig. 3

Other catalytic poisons (P, S, Sb, Se, Te) did not exert any appreciable influence on the yield of glyoxalic acid, but in the presence of a suitable concentration of Cu^{++} ions in the solution there was a complete cessation of the reduction process, connected no doubt with the low value of the hydrogen overvoltage on the surface of the copper cathode.

Conclusions

The experimental data described above show, as was anticipated, that the catalytic poisons of the reaction $2H = H_2$ cause an increase in the yield of the reduction processes, which are based on the action of atomic hydrogen (e. g., nitrobenzene), and that they either remain without influence or lower the yield in processes of the second type, which are dependent on the value of the hydrogen overvoltage (e. g. – oxalic acid). Thus the behaviour of a given system in regard to catalytic poisons serves as a good criterion for determining whether the process belongs to the first or second type of reduction mechanism.

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CHEMISTRY

Thermal and thermogravimetric analysis of hydrated cupric chloride

by
J. KAMECKI and J. TRAU

Presented by B. KAMIEŃSKI on December 16, 1954

The problem of the thermal dehydration of hydrated cupric chloride arose in connection with the work of the authors on the processes taking place during the chloridizing roasting of copper ores.

The data on the subject of hydrates of cupric chloride remain even to this day uncertain and even contradictory. According to Sabatier [1], $CuCl_2 \cdot 2H_2O$ should melt at 110°C., and then lose its water. Boye [2] states that there are four different hydrates (testing up to 116.8°C.), so that the tetrahydrate should be transformed into trihydrate at 15.0°, the trihydrate into dihydrate at 25.78°, and the dihydrate into monohydrate at 42.2°. According to H. Benrath [3], on the other hand, there should exist only one hydrate, $CuCl_2 \cdot 2H_2O$, in the range of temperatures from 0° to 100°.

In order to remove the uncertainties as to the appearance of different hydrates of cupric chloride, and to establish the temperatures of their decomposition, a series of thermal and thermogravimetric analyses of several preparations of hydrated cupric chloride were carried out.

The thermal analyses were carried out within a range of temperatures from 68° to 260°C., or from 2° to 210°C., or from 20° to 230°C., using for the measurements of temperature a Ni-NiCr thermocouple and a self-registering Foxboro potentiometer, the scale of which ranged from 2 to 12 mV. As in a previous work of one of the authors [4], the wires of the thermocouple near the welding were wound spirally in order to increase the heat capacity of that part of the thermocouple inserted into the substance being tested. This rendered it possible to avoid errors connected with the conduction of heat through the wires of the thermocouple. The thermocouple was calibrated in a glycerine thermostat by means of standard thermometers made by Messrs. Hugo Schlegel, Ilmenau in Thür. By connecting the thermocouple serially with a constant voltage

of 3 or 2 mV. it was possible to take measurements with this apparatus within the range of 2—210°C. or of 20—230°C. (the cold ends of the thermocouple were in a Dewar flask filled with water at a temperature of 20°C.).

The powdered substance was placed in a glass test-tube about 65 mm. long and 13 mm. in diameter, and this in turn was placed in a cylindrical block of copper with an appropriate bore for the test-tube. The whole was heated in a furnace by means of a constant current. The thermal effects on the heating curves were ascertained and defined by comparison of the course of the thermal analysis curves, with the "furnace curve", i. e. the heating curve of a neutral substance under the same conditions. Chemically-pure ignited sodium chloride was used, as being a substance which does not give any thermal effects within the range of the temperatures tested.

Samples about 4 g. in weight were used for the measurements (those of about 2 g. gave less distinct peaks on the heating curve). The measurements were taken in an atmospheric pressure of 738 to 748 mm. Hg. The relative humidity of the air was from 50 to 70 per cent.

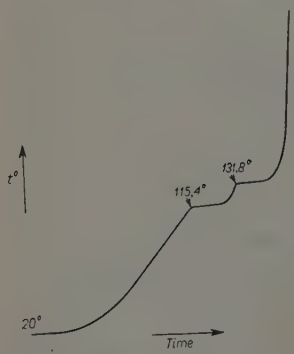

Fig. 1. Heating curve of preparation I. Rate of heating, 3.02°C./min.

For the thermal analyses there was used: I. a preparation of chemically pure hydrated cupric chloride made by the Fabryka Odczynników Chemicznych at Gliwice. The preparation was distinctly damp, of a greenish hue, and contained 30.59 per cent of water, which corresponds to the composition of $CuCl_2 \cdot 3.29\ H_2O$. The water content was determined by drying the salt to a constant weight in a drying stove at 140°C.

On the heating curves within the range of 20°—260°, two distinct endothermic effects appear at mean temperatures of 115.3°C. (individual measurements: 115.5°, 115.5°, 114.5°, 117.6°, 113.5°) and 132.2° (individual measurements: 132.5°, 132°, 132.1°, 134°, 130.4°). The rate of heating of the substance was 2.22°C. or 3.02°C. per minute. The results were confirmed by differential thermal analysis (as a reference substance, chemically pure ignited aluminium oxide was used).

One of the heating curves is shown in Fig. 1.

Preparation II was obtained by crystallizing out cupric chloride from water at a temperature under 0°C. The crystals obtained were filtered off without washing. The wet preparation, immediately after filtering, contained water in an amount corresponding to the composition $CuCl_2 \cdot 4.2\ H_2O$. The thermal analysis of this preparation from $+2$°C.

did not show any new heating effects besides those ascertained in the case of preparation I.

Preparation III was obtained by drying preparation I in air (temperature c. 20°C.; relative humidity, c. 65 per cent). The preparation, when dried in this way, contained 21.11 per cent H_2O (determined as previously) which corresponds to the content of 1.998 mols H_2O per 1 mol $CuCl_2$. The preparation was finely crystalline, friable, blue in colour.

On the heating curves of this preparation (the heating was carried out at a rate of 1.36°, 2.22°, 3.02°, 6°, and 7° per minute), only one endothermic effect was found, which corresponds strictly to the second effect observed on the heating curves of preparations I and II. The rate of heating (within the limits given above) was not seen to influence the temperature of the observed stationary point.

Similar measurements were taken using Merck's chemically pure dihydrate of cupric chloride, not containing arsenic (preparation IV). The water content amounted to 21.15 per cent., which corresponds to 2.002 mols H_2O per 1 mol $CuCl_2$. The rate of heating was 3.02°C. per minute. The results obtained were in agreement with those obtained with preparation III.

The results of all measurements have been collected in Table I, in which the kind of preparation is given in the second column, the mass of the sample in the third, the rate of heating in degrees Centigrade per minute in the fourth, and the temperature corresponding to the first and second stationary points on the heating curve in the fifth and sixth columns respectively.

TABLE I

Results of thermal analyses

No.	Kind of preparation	mass of sample in g.	rate of heating in °C. per minute	temperature of the	
				first stationary point	second stationary point
1	I	4	2.22	115.5	132.5
2	I	4	2.22	115.5	132.0
3	I	2	3.02	114.5	132.1
4	I	4	3.02	117.6	134.0
5	I	4	3.02	113.5	130.4
6	II	4	3.02	115.7	131.3
7	III	4	1.36	—	131.4
8	III	4	2.22	—	132.1
9	III	4	3.02	—	130.9
10	III	4	6.00	—	132.3
11	III	4	7.00	—	131.0
12	IV	4	3.02	—	132.0
			mean:	115.4	131.8

The thermal analyses were supplemented by observations of the changes in the preparation during the process of analysis. For this purpose the thermal analysis was carried out in an all glass apparatus (a test-tube placed in an air-bath made from a beaker with an asbestos cover, heated in a crucible with sand).

During the thermal analysis of preparation I (with excess water) only a change in the colour, from greenish to rather light greenish-yellow, was observed up to 130°. Above the temperature of 130° the substance on further heating began to turn brown, forming a dense mass. During the thermal analysis of preparation IV ($CuCl_2 \cdot 2H_2O$) no changes in the

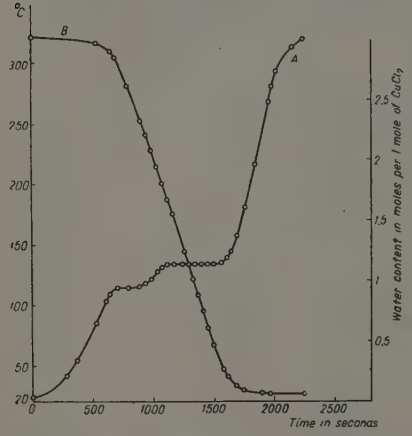

Fig. 2. Curve of thermogravimetric analysis (A) and simultaneous thermal analysis (B). Preparation I.

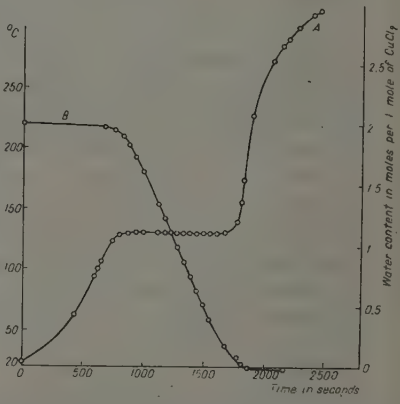

Fig. 3. Curve of thermogravimetric analysis (A) and simultaneous thermal analysis (B). Preparation IV

appearance of the preparation were observed up to a temperature of 115°. From a temperature of 116° upwards, the sample began to turn green from the top. The change in colour spread gradually over the whole sample. At a temperature of 129°, the substance began to turn brown on the walls of the test-tube. Then the temperature became stationary at 131.5°, so that the colour of the whole sample gradually changed to brown.

A number of thermogravimetric analyses were carried out in order to establish definitively the course of the reaction during thermal analysis.

For the measurements an ordinary analytical balance was used, in which one pan was replaced by an aluminium wire on which was hung the test-tube with the substance under examination, and with a slender mercury thermometer inserted into it. The test-tube was placed in a vertical electric pipe furnace. At the same time, the loss of mass of the sample (thermogravimetric analysis) and the temperature as a function

of the time of heating (thermal analysis) were measured. The results of the measurements when using preparation I are given in Fig. 2, and those when using preparation IV in Fig. 3.

The results of the thermal and thermogravimetric analyses may be presented with particular advantage in the form of a graph of the dependence of water loss in moles on the temperature. Graphs are then obtained which enable a direct reading to be made of the water loss in the given temperature.

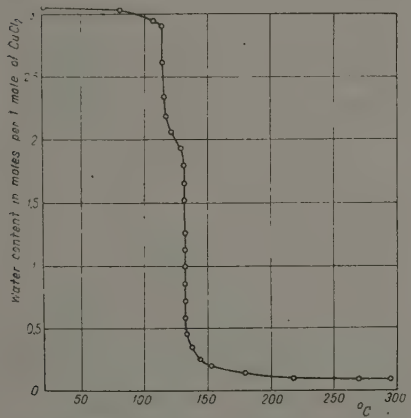

Fig. 4. Curve of dependence of water content in moles per 1 mol of $CuCl_2$ on temperature. Preparation I

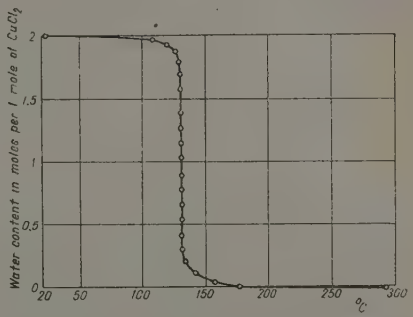

Fig. 5. Curve of dependence of water content in moles per 1 mol of $CuCl_2$ on temperature. Preparation IV

In Figures 4 and 5, such graphs are given for the measurements represented in Figs. 2 and 3.

The results of all these measurements give a uniform indication that within the range of temperatures from 2° to 131°C., there really exists only one hydrate of cupric chloride, of a composition $CuCl_2 \cdot 2H_2O$, which at a temperature of 131.8°C. completely loses both molecules of water. Preparations with a greater water content (preparations I and II) possess excess water (over 2 moles H_2O per 1 mol of $CuCl_2$) in the form of a saturated solution, covering the surface of the crystals of $CuCl_2 \cdot 2H_2O$ and giving them a greenish tint [5]. This solution boils at a temperature of 115.4° and there remains $CuCl_2 \cdot 2H_2O$.

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BULLETIN DE L'ACADÉMIE
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CHEMISTRY

Migration of components of binary liquid metallic alloys in the electric field from the point of view of thermodynamics of irreversible processes

by
B. BARANOWSKI

Presented by M. ŚMIAŁOWSKI on December 28, 1954

Besides the movement of electrons, the electric field in liquid metallic alloys also produces a certain separation of the metallic components [1]. This process takes place up to the moment when the separating action of the electric field due to isothermic diffusion is abolished [2]. So far, the kinetic [3], [4] theory has been presented and also the thermodynamic theory of transference of components of liquid metallic alloys in the electric field [5]. The latest theory is based on the so-called theory of pseudo-thermostatics [6], which means that it is based on the use of classical methods of thermodynamics applied to irreversible phenomena [7].

In this work is exposed the theory of this phenomenon in binary liquid alloys, based on Onsager's thermodynamics of irreversible processes [6]. A detailed description will be given in another publication [8].

Phenomenological equations

At constant temperature and with the exclusion of all external forces except the electrical, the flows in binary alloys can be given in the following form [8]:

$$\begin{aligned} \mathbf{J}_1 &= L_{11}\left(\mathbf{X}_1 - \frac{v_1}{v_2}\mathbf{X}_2 + e_1^*\mathbf{X}_e\right) \\ \mathbf{J}_e &= L_{11}e_1^*\left(\mathbf{X}_1 - \frac{v_1}{v_2}\mathbf{X}_2\right) + L_{ee}\mathbf{X}_e, \end{aligned} \tag{1}$$

where $\mathbf{J}_1, \mathbf{J}_e$ represent the flows of one of the metallic components of the alloy and the electrons expressed in moles per unit of surface and time, L_{11}, L_{ee} express the phenomenological coefficients, $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_e$ — the thermodynamic forces corresponding to both components and the electrons, v_1, v_2 — the partial volumes of both components, e_1^* is the flow of electrons connected with the unit flow of component 1 by vanishing

thermodynamical force corresponding to the electrons ($\mathbf{X}_e = 0$). Equation (1) has been obtained under the assumption of vanishing volume flow of the alloy:

$$\mathbf{J}_1 v_1 + \mathbf{J}_2 v_2 = 0, \tag{2}$$

where $\mathbf{J}_2$ is the flow of the second component.

Assuming that one of the components has a considerable ascendancy over the other, the phenomenological equations can be given in the following form:

$$\begin{aligned} \mathbf{J}_1 &= L_{11}(-\operatorname{grad}\mu_1 - E^* \operatorname{grad}\varphi) \\ \mathbf{J}_e &= L_{11}\epsilon_1^*(-\operatorname{grad}\mu_1 - \Delta e \operatorname{grad}\varphi) + L_{ee}\operatorname{grad}\varphi, \end{aligned} \tag{3}$$

where μ_1 is the chemical potential of the dissolved component, φ the electrical potential and F Faraday's number.

The values of E^* and Δe are obtained from the following relations:

$$\begin{aligned} E^* &= e_1 - \frac{v_1}{v_2} e_2 - e_1^* F \\ \Delta e &= e_1 - \frac{v_1}{v_2} e_2, \end{aligned} \tag{4}$$

where e_1 and e_2 express the effective molecular charges of ions of both components of the alloy.

Phenomenological coefficients

The phenomonological coefficients in equations (1) and (3) can be approximately expressed by using the definitions of the transfer number (t_1):

$$t_1 = \left(\frac{\mathbf{J}_1}{\mathbf{J}_e}\right) \operatorname{grad}\mu_1 = 0 \tag{5}$$

and of the specific electric conductivity $\varkappa$

$$\Delta e \, \mathbf{J}_1 - F \mathbf{J}_e = -\varkappa \operatorname{grad}\varphi. \tag{6}$$

Equation (6) is also valid by the additional condition: grad $\mu_1 = 0$. The application of the last two equations gives the following approximate expressions for the phenomenological coefficients:

$$L_{11} = -\frac{\varkappa t_1}{F E^*} \tag{7}$$

$$L_{ee} = \frac{\varkappa}{F^2}. \tag{8}$$

Maximum separation of alloy components in the electric field

Applying equation (3) to the case when

$$\mathbf{J}_e \neq 0; \ \mathbf{J}_1 = 0; \ \operatorname{grad}\varphi \neq 0, \tag{9}$$

we obtain an expression for the maximum separation of the components of the alloy in a given electric field:

$$\text{grad}\,\mu_1 = -E^* \,\text{grad}\,\varphi. \tag{10}$$

As shown by Drakin [5] a formula of this type has been proved experimentally. His expression [5] is obtained as a special case of expression (10) on the assumption that

$$v_1 = v_2; \qquad e_1^* = 0 \tag{11}$$

and also expressing the chemical potential for an ideal system.

Putting e_1^* equal to zero means not taking into account the influence of the electron movements on the movements of the metallic ions.

Diffusion potential

In the system under discussion, the isothermic diffusion is accompanied by the appearance of the electric field, the diffusion potential.

We obtain an expression for this value by applying equation (3) to the stationary state:

$$\mathbf{J}_1 \neq 0; \qquad e_1 \mathbf{J}_1 + e_2 \mathbf{J}_2 - F\mathbf{J}_e = 0. \tag{12}$$

Using these conditions we obtain

$$\text{grad}\,\varphi = \frac{t_1}{F}\,\text{grad}\,\mu_1. \tag{13}$$

Expressing the chemical potential for the ideal system we obtain from formula (13) an expression identical to that derived previously by Schwartz using the pseudo-thermostatic method [9], [7]. This formula has frequently been proved experimentally.

Connection between the diffusion coefficient and the values E^*, Δe and t_1

Expressing the flow of mass in the isothermic diffusion first with the help of Fick's law and then by equation (3), a connection between the isothermic diffusion coefficient D, the transference number and the values E^* and Δe can be obtained

$$D = \frac{\varkappa t_1 RT\left(1 + \dfrac{d\ln f_1}{d\ln c_1}\right)\cdot\left(\dfrac{E^* t_1}{F} - 1\right)}{c_1 E^* (t_1 \Delta e - F)}, \tag{14}$$

where R is the gas constant, T — the absolute temperature, f_1 — the activity coefficient and c_1 the concentration.

The above formula can be simplified by taking into account that the product $t_1 \Delta e$ is small as compared with the Faraday number and that the expression $E^* t_1/F$ is negligible in comparison to unity. For-

mula (14) is then identical with that derived by Schwarz [4], assuming $c_1^* = 0$ and the ideality of the system:

$$D = \frac{\varkappa t_1 RT}{c_1 \Delta e F}. \tag{15}$$

Assuming further equal partial volume an expression of Drakin's theory is obtained [5].

I exrpess my thanks to Professor K. Gumiński for supervising this work.

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BULLETIN DE L'ACADÉMIE
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An attempt to mechanize the graniformameter — a device for the separation of gravel grains according to rounding grade

by

B. KRYGOWSKI

Presented by S. LESZCZYCKI on November 3, 1954

In 1937 the author published an article in the Archives of Mineralogy [1], dealing with a device for sorting sand or gravel grains according to their shape and roundness. This machine was based on the principle of an inclined plane which, when fixed at various angles, sorted the samples of gravel grains, which were poured onto it, into rounded, subangular and angular grains. The round grains rolled down when the inclination of the plane was very slight. They were thus separated from the remaining grains of that sample, which remained on the inclined plane, because its inclination was still too small and they could not roll down. With a growing angle of inclination of the plane, the subangular grains rolled down and so did finally the angular ones. This method was very simple and considerably increased the speed of determining the shape of gravel and sand grains. This determination had so far been carried out by means of the binocular microscope. Although it was unusually toilsome, slow and subjective, it was practically the only one in use.

In 1940, Preobrajenski from Kiev took an interest in this device and made some improvements in it [2]; in Switzerland attempts were also made in this direction. All this (see also [3], [4], [5]), encouraged the author to continue the study of the graniformameter, viz. the trend of its full mechanization. The enclosed illustrations represent a new, considerably mechanized model of this machine.

In previous machines [1] the inclined plane had been made of a frosted glass plate which could be adjusted if necessary at any angle. In the new model a driving-belt of rubber forms the inclined plane — *a*; it is placed upon two cylinders — *b*, which, when set in motion by an electric drive — *c*, bring about the shifting of the belt upwards. The belt,

the cylinders and the frame which encloses them form a common element, i. e., the inclined plane, which can be fixed at any angle. In the new model the inclined plane (in this case the rubber belt) is mobile. It rotates on the cylinders; this was not the case in the former model. In the lower part of the inclined plane there is a rubber roll — *e*, which revolves in the same direction as the driving-belt. This roll and the rubber belt form a nest into which is poured the sample of gravel to be

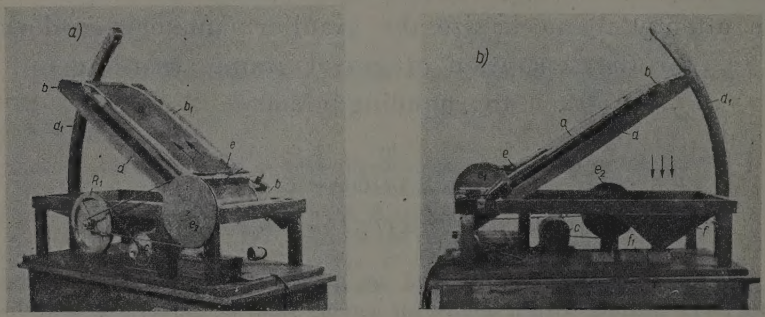

Fig. 1 (a. b.) — Model of mechanized graniformameter (two pictures of the same device, showing almost all its details).
a — inclined plane in the shape of a rubber driving-belt. The double arrow on the belt indicates the direction of its motion.
b — wooden roll on which the rubber belt revolves.
b_1 — tapes keeping the grains from falling off at the sides. They are stretched by means of weights — b_2.
c — small electric motor
c_1-c_2 — driving wheels carrying the electric drive into the cylinder
d — frame composing, together with the cylinder and the rubber belt, a common element — the inclined plane.
d_1 — ledge — a goniometer on which rests the inclined plane — *a*, fixed at the desired angle.
e — rubber cylinder, rotating in the same direction as the rubber belt. It produces a fold — the nest, into which the treated grains are poured.
f — cardboard reservoir, funnel shaped, into which fall the grains, selected by the inclined plane. The direction of spilling of the grains is shown by a triple arrow. (Fig. 1b)
f_1 — small boxes into which the selected grains drop.

classified. When the inclined plane is set in motion and the maximum angle is fixed at 45°, a sample of gravel is poured into the nest, e. g. 200 to 400 grains (or a certain weight), the diameter of which varies from 3 to 4 mm. The rubber belt picks out from the nest the most angular grains that have the shape of plates or discs, and casts them into the box *f*, beyond the upper roll. After two minutes, without stopping the machine, the inclined plane is set at an angle of 40°, then 35°, 30°, 25°, etc. until no grains are left in the nest — *e*. When the inclinations are considerable, the angular grains are separated. The inclined plane (rubber belt) lifts

them and throws them into the box. When the inclinations are smaller, only the strongly rounded grains remain in the nest, e. g. at 2.5°. The grains that are almost spherical are thrown into the box only when the inclination is 0°. The inclined plane can be set at various angles, e. g. every 5° or 2.5°, as required. The inclination may be changed every minute or every two minutes; this depends, likewise, upon the type of study and requirement.

The sample of sand which by means of the binocular was segregated in the course of 1 to 2 hours, was selected by the graniformameter within 10 to 15 minutes; subjectivism is, moreover, completely eliminated. Persons working with the binocular method selected the same sample differently, since by sight the estimate of the rounding-grade of the grain is very deceptive.

The table shows the analysis of gravel from granite weathering products on Mount Śnieżka, the grains of which are from 3 to 4 mm. in diameter, and of Poznań outwash sands. These analyses were made by means of the graniformameter described here.

Angle of inclination of the plane (classes of grains)	Average percentage of grains selected at a given inclination:	
	Granite weathering products from Mount Śnieżka	Poznań outwash sands (Główna)
45°	0.25	0.00
40°	1.72	0.33
35°	3.17	0.97
30°	21.20	4.52
22°	67.96	22.16
20°	5.70	66.92
15°	0.00	5.10

It is obvious that in both cases the bulk of the gravel consists mainly of two classes of grains; for the granite weathering products — the 25° and the 30° classes, for the outwash gravel — the 20° and the 25° classes. These classes of grains are typical and constant for the aforesaid materials. The difference between both materials is evident. In the granite weathering products the grains of the 25°-class are the most numerous, while in the outwash gravel those of the 20°-class form the major part. This means that the grains of the granite weathering products which were not subjected to transport and in connection therewith to any rubbing off, are more granular than the grains of outwash gravel which during the long transport undergo a slight rolling and rounding of the edges. Hence, while the sharp angular grains of the granite wea-

thering products were picked out of the nest at the greater inclination of 25-30°, the outwash gravel grains with blunted edges were picked out already at the smaller one of 20-25°.

The device segregates grains not only according to the degree of their sphericity, but also gives information on the degree of their rolling and rubbing off.

The round grains, e. g. those from sand dunes, remain in the nest the longest. They are picked out at the smallest inclinations of 2.5° and 5°.

This description is only an account of the progress in the mechanization of the graniformameter, and does not contain closer details of the method or numerous examples of the application of the device to the analysis of various sands and gravels. These will be published in Czasopismo Geograficzne or Archiwum Mineralogiczne.

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